Entanglement Localisation Via Classical Mediator

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Contents

"You may encounter many defeats, but you must not be defeated. In fact, it may be necessary to encounter the defeats, so you can know who you are, what you can rise from, how you can still come out of it."

Maya Angelou

"Making your mark on the world is hard. If it were easy, everybody would do it. But it's not. It takes patience, it takes commitment, and it comes with plenty of failure along the way. The real test is not whether you avoid this failure, because you won't. It's whether you let it harden or shame you into inaction, or whether you learn from it; whether you choose to persevere."

Barack Obama

"A failure is not always a mistake. It may simply be the best one can do under the circumstances. The real mistake is to stop trying."

B. F. Skinner

"It does not matter how slowly you go as long as you do not stop."

Confucius

"Success is the sum of small efforts, repeated day in and day out."

Robert Collier

"Success is not measured by what you accomplish, but by the opposition you have encountered, and the courage with which you have maintained the struggle against overwhelming odds."

Orison Swett Marden

"Never limit yourself because of others' limited imagination; never limit others because of your own limited imagination."

Mae Jemison

"No matter what you're going through, there's a light at the end of the tunnel and it may seem hard to get to it but you can do it and just keep working towards it and you'll find the positive side of things."

Demi Lovato

"Few things in the world are more powerful than a positive push. A smile. A world of optimism and hope. A 'you can do it' when things are tough."

Richard M. DeVos

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2 Abstract

Firstly, with reference to Ref. [8], I understand that Prof Tomasz and his team has came out with the theoretical model as shown below in Fig. 1 to investigate whether an inaccessible object can be used to increase quantum entanglement between two remote agent particles that individually interact with the inaccessible object but not directly coupled to each other. With intuition from this reference and using it as a base, in this thesis, we will first look into this theoretical model where we investigate the entanglement distribution between two particles A and B via continuous interactions locally with single particle C. Particles A and B are located in two separated laboratories, which are operated by agents called Alice and Bob, where particle A is in Alice's lab and particle B is in Bob's lab. Particle A interacts locally with C and particle B interacts locally with C but A and B does not interact directly. Therefore, the total Hamiltonian is $H_{AC} + H_{BC}$ and entanglement between A and B increases although by all quantifiers of classicality C remains classical thoughout the evolution. However, as we know that entanglement does not grow via local operations and classical communication, then one might wonder how is this theoretical model compatible to this statement. The answer is that there is entanglement in the tripartite ABC system already at the beginning.

Looking at this theoretical model, we work out several equations and steps to measure the negativity between A and B which indicates entanglement. We were able to analytically prove that there is indeed entanglement between A and B where it oscillates from 0 to $\frac{1}{2}$.

Moving on from that, since the phenomenon of the above theoretical example has never been observed in a lab, we then make calculation within physically motivated model to check for the feasibility of the demonstration. We consider two electron spins A and B which are confined in separate quantum dots. Our idea is to utilize the hyperfine interactions between the electron spins and the spins of the nuclei at the atoms forming the quantum dots with the environment for entanglement localisation. Therefore, we then came up with a toy model where only one spin is coupled to the environment to measure the entanglement. We assume that spin B is not coupled to any outside spins and spin A has Heisenberg interaction with its single-spin environment A'. Working through many cases, we found that in the limit of pure dephasing, starting with a suitable initial state, entanglement can indeed be localised via interactions with classical local environment.

3 Introduction

Quantum entanglement is a form of correlations between quantum particles that does not increase under local operations and classical communication [1]. In the simplest case it involves two particles in separated laboratories, which are operated by agents usually called Alice and Bob: particle A is in Alice's lab and particle B in Bob's. According to the definition given, if A and B are not entangled, entanglement between them cannot be created by exchanging (other) classical particles or information.

In this thesis we model classical communication between the laboratories by a single particle C which is continuously coupled to A and B. In order to preserve local character of interactions we assume that A and B are not directly coupled, i.e. the total three-particle Hamiltonian is of the form $H_{AC} + H_{BC}$, see Fig. 1. We will provide example where entanglement between A and B increases although by all quantifiers of classicality C remains classical thoughout the evolution. This sounds contradictory to the very definition of entanglement, but we resolve the tension by showing that generated $A : B$ entanglement is always smaller than initial entanglement bewteen A and BC together. Therefore, the whole process could be named as "entanglement localisation" from partition $A : BC$ to subsystem $A : B$ only. We emphasise the counter-intuitive fact that this entanglement localisation can happen via classical mediator C.

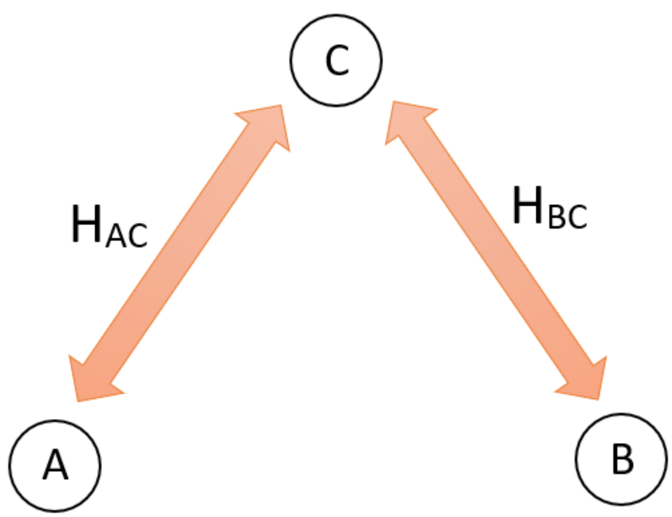

Figure 1: Objects A and B individually interact with a mediator C, but not with each other. The interactions are described by Hamiltonians H_{AC} and H_{BC}. In this thesis all systems A, B and C are two-dimensional, i.e. qubits.

4 Background

4.1 Pure and Mixed States

A quantum system whose state is known exactly is said to be in a pure state. We denote the pure state as $|\psi\rangle$ and its density matrix as $\rho = |\psi\rangle\langle\psi|$. Mixed state, on the other hand, is a mixture of different pure states $|\psi_i\rangle$ and its density operator is $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, where p_j are probabilities. Furthermore, we need to take note that the different pure states $|\psi_i\rangle$ mentioned above may not be orthonormal to each other, hence we need to represent ρ in diagonal representation so that the different pure states can form orthonormal basis and the new probabilities generated will fall into the diagonal representation.

By calculating trace $\text{tr}(\rho^2)$, we are able to determine whether a state is pure or mixed. When $\text{tr}(\rho^2)$ gives 1, this shows that ρ is a pure state. Otherwise, when its value is less than 1, it reveals a mixed state [2]. For more information of how the trace can be calculated, one may refer to the Appendix section below.

4.2 Separability for Pure and Mixed States

In this section, we will be looking at what separability means in bipartite system as well as tripartite system for pure and mixed states. This is rather useful to our further calculations in this report.

4.2.1 Separability for Pure States

Suppose the quantum state of a bipartite system XY is pure, if the state can be written in this form:

$$|\psi_{xy}\rangle = |\psi_x\rangle \otimes |\psi_y\rangle, \tag{1}$$

Then we can say that the state is separable. If not, then entangled.

For a quantum state of tripartitie system XYZ, separability can come in two forms, it can be completely separable or bi-separable.

Suppose the quantum state for tripartite system XYZ is pure and if it is completely separable, it can be written in this form:

$$|\psi_{xyz}\rangle = |\psi_x\rangle \otimes |\psi_y\rangle \otimes |\psi_z\rangle. \tag{2}$$

This equation also shows that all X, Y and Z states are definite states and they behave locally as three completely independent subsystems [2].

On the other hand, suppose the quantum state for tripartite system XYZ is pure and if it is bi-separable, it can be written in this form:

$$|\psi_{xyz}\rangle = |\psi_{xy}\rangle \otimes |\psi_z\rangle. \tag{3}$$

This equation shows that the state XY and Z are pure and independent, however there is entanglement existing between subsystem X and Y [2]. Furthermore, for bi-separable states, it can come in many forms, for instance one example is Eq. (3). Others can be as follows,

$$|\psi_{xyz}\rangle = |\psi_{xz}\rangle \otimes |\psi_{y}\rangle, \tag{4}$$

$$|\psi_{xyz}\rangle = |\psi_{x}\rangle \otimes |\psi_{yz}\rangle. \tag{5}$$

Hence, one must indicate clearly the state is separable in which partition. For example, Eq. (3) can be written as $XY : Z$ separable, Eq. (4) can written as $XZ : Y$ separable and lastly Eq. (5) can be written as $X : YZ$ separable.

4.2.2 Separability for Mixed States

Suppose the quantum state of a bipartite system XY is mixed, if the state can be written in this form:

$$\rho_{xy} = \sum_i \lambda_i |\psi_i^x\rangle\langle\psi_i^x| \otimes |\psi_i^y\rangle\langle\psi_i^y| \tag{6}$$

$$= \sum_i \lambda_i \rho_i^x \otimes \rho_i^y \tag{7}$$

where λ_i are probabilities.
Then we can say that the state is separable. If not, then entangled.

For a quantum state of tripartitie system XYZ, similar to pure states, separability can come in two forms, it can be completely separable or bi-separable.

Suppose the quantum state for tripartite system XYZ is mixed and if it is completely separable, it can be written in this form:

$$\rho_{xyz} = \sum_i \lambda_i |\psi_i^x\rangle\langle\psi_i^x| \otimes |\psi_i^y\rangle\langle\psi_i^y| \otimes |\psi_i^z\rangle\langle\psi_i^z| \tag{8}$$

$$= \sum_i \lambda_i \rho_i^x \otimes \rho_i^y \otimes \rho_i^z \tag{9}$$

For bi-separable states, similarly it can be separable in three partitions, for instance we may look at $XY : Z$ separable, it can be written in this form:

$$\rho_{xyz} = \sum_i \lambda_i |\psi_i^{xy}\rangle\langle\psi_i^{xy}| \otimes |\psi_i^z\rangle\langle\psi_i^z| \tag{10}$$

$$= \sum_i \lambda_i \rho_i^{xy} \otimes \rho_i^z \tag{11}$$

4.3 Negativity as Entanglement Quantifier

In this thesis we only use negativity to measure entanglement. Negativity was introduced in Ref. [3] and it is defined for a bipartite state as follows:

$$N_{X:Y}(\rho) = \sum_j |\lambda_j^-|, \tag{12}$$

where λ_j^- are negavite eigenvalues of the matrix obtained by partial transposition of ρ. Partial transposition can be done on any subsystem, X or Y, the final result is the same. We note that subsystem X can be composed of many particles. For example, in a tripartite state ρ_{ABC} one can discuss bipartite entanglement between A and BC together, so $X = A$ and $Y = BC$.

Here is how partial transposition of ρ works. First we have our ρ,written in some orthonormal bases for subsystems X and Y:

$$\rho = \sum_{X,Y,X',Y'=0,1} \rho_{XY,X'Y'} |XY\rangle\langle X'Y'|. \tag{13}$$

Doing partial transposition on X means

$$\rho_{XY,X'Y'} \longrightarrow \rho_{X'Y,XY'} \tag{14}$$

where we switch X and X'.
Hence we get,

$$\rho^{Tx} = \sum_{X,Y,X',Y'} \rho_{X'Y,XY'} |XY\rangle\langle X'Y'|. \tag{15}$$

Lastly, negative eigenvalues are obtained through finding determinant of the matrix through the formula: $det(\rho^{Tx} - I\lambda) = 0$.

Furthermore, separable states (not entangled states) give rise to positive eigenvalues after partial transposition. One can prove it by starting with the general form of separable state,

$$\rho = \sum_i \mu_i \rho_{Xi} \otimes \rho_{Yi}, \tag{16}$$

where μ_i are probabilities.
The partial transpose, say with respect to X, gives

$$\rho^{T_x} = \sum_i \mu_i (\rho_{Xi})^T \otimes \rho_{Yi}. \tag{17}$$

Now, any density matrix can be diagonalized,

$$\rho_{Xi} = \sum_j \eta_j |j\rangle\langle j|, \tag{18}$$

where $|j\rangle$ form diagonal basis and η_j are new probabilities (eigenvalues).

From this, one can see that ρ_{Xi}^T is still a valid density matrix. So, $(\rho_{Xi})^T \otimes \rho_{Yi}$ is also a valid density matrix for X and Y. The mixture $\sum_i \mu_i (\rho_{Xi})^T \otimes \rho_{Yi}$ will therefore also be a valid density matrix, ie. it has positive eigenvalues.

4.4 Discord as Nonclassicality Quantifier

Quantum discord is a measure of nonclassical correlations between two subsystems of a quantum system [4–6]. It is simplest to introduce discord by giving a class of states for which it vanishes:

$$\chi = \sum_y p_y \rho_{x|y} \otimes |y\rangle\langle y|, \tag{19}$$

where it is important that the states $\{|y\rangle\}$ are orthogonal. For this class of states subsystem Y could therefore be measured in the basis $\{|y\rangle\}$ and this measurement (when averaged over all measurement results) does not modify the state χ. In other words vanishing discord implies that there is a local measurement which does not modify (perturb) the whole measured system.

Discord can then be introduced as a distance to a set of states in Eq. (19), see e.g. [7]. It is clear from this description that discord is not a symmetric quantity. We denote by $D_{X|Y}$ discord measured as a distance to the set of states in Eq. (19), where the states of subsystem Y are orthogonal, and by $D_{Y|X}$ discord measured as a distance to an analoguous set, where this time the states of subsystem X are orthogonal.

5 Entanglement Localisation Via Classical Mediator

In this section we review perhaps the simplest example of this phenomenon, given in Ref. [8]. Consider three qubits (two-level systems) in the following initial state:

$$\rho_0 = \frac{1}{2}|\psi_+\rangle\langle\psi_+| \otimes |+\rangle\langle+| + \frac{1}{2}|\phi_+\rangle\langle\phi_+| \otimes |-\rangle\langle-|, \tag{20}$$

where the order of parties is ABC and the states being mixed are as follows:

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \tag{21}$$

$$|\phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{22}$$

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle). \tag{23}$$

The first two states are so called Bell states and they describe particles A and B. The last line gives the states of C. This initial state has the following entanglement properties:

$$N_{A:B}(\rho_0) = 0, \qquad N_{A:BC}(\rho_0) = N_{AC:B}(\rho_0) = \frac{1}{2}, \tag{24}$$

which are easy to verify by direct computation.

Consider now the following Hamiltonian coupling A and B through the mediator C:

$$H = \hbar\omega(\sigma_A^X \otimes I_B \otimes \sigma_C^X + I_A \otimes \sigma_B^X \otimes \sigma_C^X), \tag{25}$$

where I is the 2×2 identity matrix, σ^X stands for Pauli x matrix and the index clarifies on which subsystem the matrix is acting. The interaction energy of every pair of subsystems is set to $\hbar\omega$. Clearly, the states $|\pm\rangle$ of C are the eigenstates of the Hamiltonian and hence they are stationary. At all times system C is in one of these two orthogonal states and we have:

$$\rho_t = \frac{1}{2}|\psi_t\rangle\langle\psi_t| \otimes |+\rangle\langle+| + \frac{1}{2}|\phi_t\rangle\langle\phi_t| \otimes |-\rangle\langle-|. \tag{26}$$

Explicit calculation shows:

$$|\psi_t\rangle = \cos(2\omega t)|\psi_+\rangle - i\sin(2\omega t)|\phi_+\rangle, \tag{27}$$

$$|\phi_t\rangle = \cos(2\omega t)|\phi_+\rangle + i\sin(2\omega t)|\psi_+\rangle. \tag{28}$$

In this calculation we used the fact that the total unitary dynamics operator $U = \exp(-\frac{i}{\hbar}Ht)$ can be written as $U = U_{AC}U_{BC}$ and each of these operators is given by

$$U_{AC} = \exp(-i\omega(\sigma_A^X \otimes I \otimes \sigma_C^X)) = \cos(\omega t)I - i\sin(\omega t)\sigma_A^X \otimes I \otimes \sigma_C^X, \tag{29}$$

$$U_{BC} = \exp(-i\omega(I \otimes \sigma_B^X \otimes \sigma_C^X)) = \cos(\omega t)I - i\sin(\omega t)I \otimes \sigma_B^X \otimes \sigma_C^X. \tag{30}$$

Since the states of C are orthogonal at all time there is no quantum discord throughout the evolution:

$$D_{AB|C}(\rho_t) = 0. \tag{31}$$

One also finds:

$$N_{A:B}(\rho_t) = \frac{\sqrt{1-\cos(8\omega t)}}{2\sqrt{2}}, \qquad N_{A:BC}(\rho_t) = \frac{1}{2}. \tag{32}$$

The dynamics of entanglement in AB subsystem is plotted in Fig. 2.

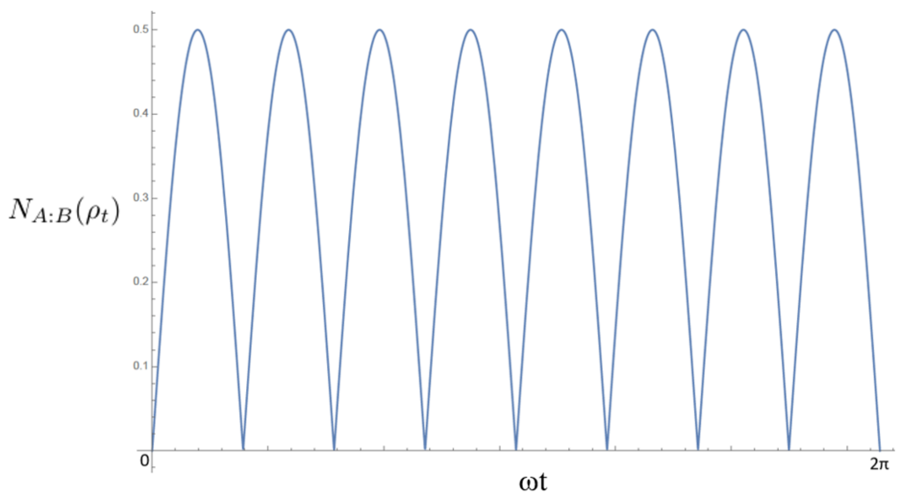

Figure 2: This figure shows a plot of $N_{A:B}$ against ωt, where we let ωt range from 0 to 2π. The plot shows entanglements between A and B. It oscillates from 0 (no entanglement) to $\frac{1}{2}$, which corresponds to maximally entangled state. At all times system C is in one of two orthogonal states, hence classical.

6 Original Calculations

In the previous section we have given an example where quantum entanglement is localised into a subsystem via classical mediator. Since this phenomenon has never been observed in a laboratory, we now make calculations within physically motivated model to check for the feasibility of the demonstration.

We consider two electron spins A and B which are confined in separate quantum dots. Quantum dots received considerable theoretical and experimental attention, see Refs. [10–15] for reviews, which result in effective techniques for the initialization, manipulation and readout of the spin state [9]. However, the coherent evolution of spin states suffers from the destructive effects of the hyperfine interaction between the electron spins and the spins of the nuclei at the atoms forming the quantum dots [9]. Our idea is to utilize these interactions with the environment for entanglement localisation. We first calculate entanglement dynamics in a toy model where only one spin is coupled to the environment and we start with the state ρ_0 given in Eq. (20).

6.1 The Toy Model

We assume that spin B is not coupled to any outside spins and spin A has Heisenberg interaction with its single-spin environment A', see Fig. 3.

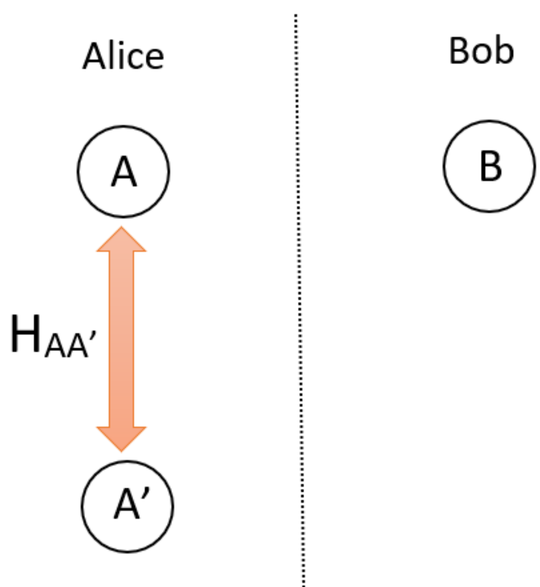

Figure 3: This figure shows spin B is not coupled to any outside spins and spin A has Heisenberg interaction with its single-spin environment A'.

Hence the interaction Hamiltonian is

$$H = -\alpha\sigma_A^Z - \alpha\sigma_B^Z + \beta\sigma_A^Z\sigma_{A'}^Z + \frac{\beta}{2}(\sigma_A^+\sigma_{A'}^- + \sigma_A^-\sigma_{A'}^+), \tag{33}$$

where $\alpha = g\mu_B B$, g is the effective electron g factor, μ_B is the Bohr magneton, B is the applied magnetic field along z axis and β is the coupling constant of the hyperfine interaction.

We will now look into three "extreme" cases below.

6.1.1 Low Magnetic Field ($\alpha \ll \beta$)

In this limit, the total hamiltonian H is effectively given by the last two terms in Eq. (33),

$$H = \beta\sigma_A^Z\sigma_{A'}^Z + \frac{\beta}{2}(\sigma_A^+\sigma_{A'}^- + \sigma_A^-\sigma_{A'}^+), \tag{34}$$

where

$$\sigma^{\pm} = \sigma^X \pm i\sigma^Y. \tag{35}$$

By simplifying Eq. (34), we get

$$H = \beta(\sigma_A^X\sigma_{A'}^X + \sigma_A^Y\sigma_{A'}^Y + \sigma_A^Z\sigma_{A'}^Z), \tag{36}$$

where

$$\sigma^X = |0\rangle\langle 1| + |1\rangle\langle 0|, \tag{37}$$

$$\sigma^Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|, \tag{38}$$

$$\sigma^Z = |0\rangle\langle 0| - |1\rangle\langle 1|. \tag{39}$$

For this calculation, we start with the initial state given in the Eq. (20). From evolution, we found the following unitary operator:

$$\begin{aligned} U = e^{-\frac{iHt}{\hbar}} &= e^{-\frac{it\beta}{\hbar}(-3)}|\psi_-\rangle_{AA'}\langle\psi_-| + e^{-\frac{it\beta}{\hbar}(1)}|\psi_+\rangle_{AA'}\langle\psi_+| \\ &+ e^{-\frac{it\beta}{\hbar}(1)}|\phi_+\rangle_{AA'}\langle\phi_+| + e^{-\frac{it\beta}{\hbar}(1)}|\phi_-\rangle_{AA'}\langle\phi_-| \end{aligned} \tag{40}$$

where we used the fact that the Bell states are the eigenstates of Hamiltonian in Eq. (36).

Using the unitary operator U found above, we further calculate the following:

$$\begin{aligned} U|\psi_+\rangle|+\rangle &= \frac{1}{2\sqrt{2}}[-e^{\frac{i3\beta t}{\hbar}}|\psi_-\rangle|0\rangle + e^{\frac{i3\beta t}{\hbar}}|\psi_-\rangle|1\rangle + e^{\frac{i\beta t}{\hbar}}|\psi_+\rangle|0\rangle + e^{\frac{i\beta t}{\hbar}}|\psi_+\rangle|1\rangle \\ &+ e^{\frac{i\beta t}{\hbar}}|\phi_+\rangle|1\rangle + e^{\frac{i\beta t}{\hbar}}|\phi_+\rangle|0\rangle + e^{\frac{i\beta t}{\hbar}}|\phi_-\rangle|1\rangle - e^{\frac{i\beta t}{\hbar}}|\phi_-\rangle|0\rangle] = |\zeta_t\rangle, \end{aligned}$$

$$U|\phi_+\rangle|-\rangle = \frac{1}{2\sqrt{2}}[-e^{\frac{i3\beta t}{\hbar}}|\psi_-\rangle|0\rangle - e^{\frac{i3\beta t}{\hbar}}|\psi_-\rangle|1\rangle - e^{\frac{i\beta t}{\hbar}}|\psi_+\rangle|0\rangle + e^{\frac{i\beta t}{\hbar}}|\psi_+\rangle|1\rangle$$
$$+ \; e^{\frac{i\beta t}{\hbar}}|\phi_+\rangle|0\rangle - e^{\frac{i\beta t}{\hbar}}|\phi_+\rangle|1\rangle + e^{\frac{i\beta t}{\hbar}}|\phi_-\rangle|0\rangle + e^{\frac{i\beta t}{\hbar}}|\phi_-\rangle|1\rangle] = |\eta_t\rangle.$$

Thus, we have the density matrix at time t,

$$\rho_t = \frac{1}{2}|\zeta_t\rangle\langle\zeta_t| + \frac{1}{2}|\eta_t\rangle\langle\eta_t| \tag{41}$$

By tracing out A' from Eq. (41), one gets ρ_t^{AB} from which entanglement between A and B can be calculated. One can also calculate entanglement in the partition $A : BA'$ from Eq. (41) by partially transposing system A. The results are plotted in Fig. 4 below, where ω is given by $\frac{\beta}{\hbar}$. We can see that $N_{A:B}$ grows but one can verify that A' is also entangled.

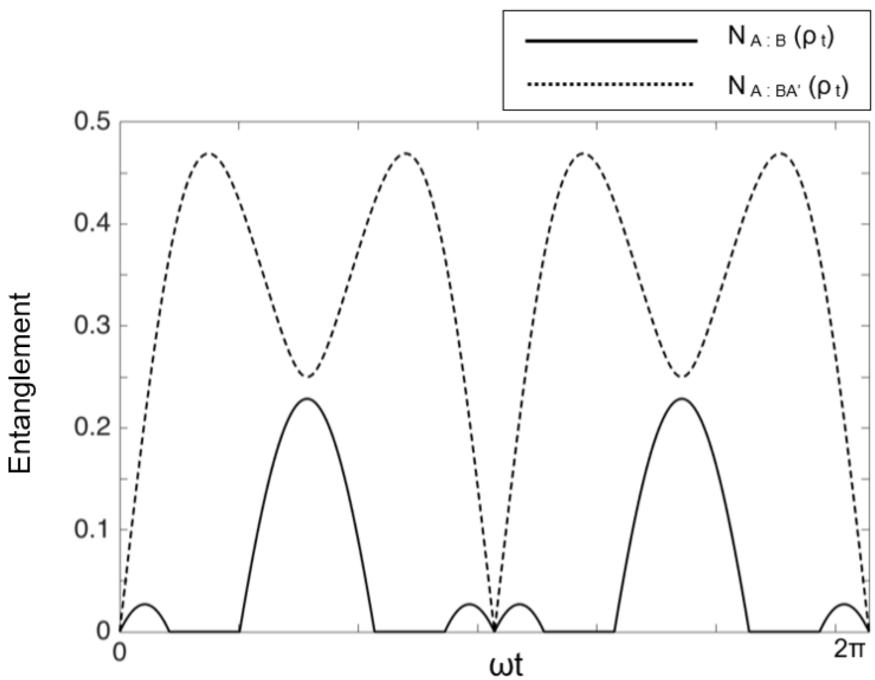

Figure 4: Plot of $N_{A:B}(\rho_t)$ and $N_{A:BA'}(\rho_t)$ against ωt, where we let ωt range from 0 to 2π.

6.1.2 High Magnetic Field ($\alpha \gg \beta$)

In this case, the total hamiltonian H is effectively given by the first two terms in the Eq. (33),

$$H = -\alpha\sigma_A^Z - \alpha\sigma_B^Z. \tag{42}$$

One can see that these terms are local and therefore do not create entanglement. Recall that entanglement cannot grow via local operations and classical communication [1]. The corresponding unitary operator is given by

$$U = U_A \otimes U_B \otimes \hat{1}_{A'}, \tag{43}$$

where e.g. $U_A = \exp(-i\alpha t \sigma_A^Z/\hbar)$. The right hand side terms are local unitary operators that keep entanglement constant (cite), i.e. not only it does not grow, it also does not decay. For example, starting with any intial state ρ, one can calculate initial entanglement $N_{A:B}$ and $N_{AB:A'}$. The corresponding evolution will result in constant entanglement. In fact, entanglement in other partitions $N_{A:A'}, N_{B:A'}, N_{A:BA'}$ and $N_{B:AA'}$ is also constant.

6.1.3 Pure Dephasing

In this case, the total hamiltonian H is effectively given by the first three terms in the Eq. (33),

$$H = -\alpha\sigma_A^Z - \alpha\sigma_B^Z + \beta\sigma_A^Z\sigma_{A'}^Z. \tag{44}$$

We start with an initial state,

$$\rho_0 = \frac{1}{2}|\psi_1\rangle\langle\psi_1| \otimes |0\rangle\langle 0| + \frac{1}{2}|\psi_2\rangle\langle\psi_2| \otimes |1\rangle\langle 1|, \tag{45}$$

where

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), \tag{46}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle). \tag{47}$$

Note that all the terms in the total hamiltonian H commute, hence the evolution is given by

$$U = e^{i\frac{\alpha\sigma_A^Z t}{\hbar}} e^{i\frac{\alpha\sigma_B^Z t}{\hbar}} e^{-i\beta\frac{\sigma_A^Z\sigma_{A'}^Z t}{\hbar}} \tag{48}$$

We know that the first two terms on the right-hand side do not create entanglement since they are coming from local Hamiltonian. Therefore, the unitary operator for creating entanglement is given by the last term:

$$U = e^{-i\beta\frac{\sigma_A^Z\sigma_{A'}^Z t}{\hbar}} = \cos\left(\frac{\beta}{\hbar}t\right)\mathbb{1} - i\sin\left(\frac{\beta}{\hbar}t\right)\sigma_A^Z\sigma_{A'}^Z \tag{49}$$

Note that this is similar to our earlier example in the Eq. (26). Here we have,

$$|\zeta_t\rangle = U|\psi_1\rangle|0\rangle = |\psi_1^t\rangle|0\rangle, \tag{50}$$

$$|\eta_t\rangle = U|\psi_2\rangle|1\rangle = |\psi_2^t\rangle|1\rangle, \tag{51}$$

where

$$|\psi_1^t\rangle = \cos\left(2\frac{\beta}{\hbar}t\right)|\psi_1\rangle - i\sin\left(2\frac{\beta}{\hbar}t\right)|\psi_2\rangle, \tag{52}$$

$$|\psi_2^t\rangle = \cos\left(2\frac{\beta}{\hbar}t\right)|\psi_2\rangle + i\sin\left(2\frac{\beta}{\hbar}t\right)|\psi_1\rangle. \tag{53}$$

We get the state at time t as shown below,

$$\rho_t = \frac{1}{2}|\psi_1^t\rangle\langle\psi_1^t| \otimes |0\rangle\langle 0| + |\psi_2^t\rangle\langle\psi_2^t| \otimes |1\rangle\langle 1|, \tag{54}$$

where $D_{AB|A'}(\rho_t) = 0$. Furthermore, by tracing out system A', we get

$$\rho_{AB}^t = \frac{1}{2}|\psi_1^t\rangle\langle\psi_1^t| + |\psi_2^t\rangle\langle\psi_2^t|. \tag{55}$$

From here, one can calculate the entanglement and the results are the same as plotted in Fig. 2, but with ω is given by $\frac{\beta}{\hbar}$. We can see that entanglement grows via classical A'.

7 Conclusions

We have reviewed the phenomenon of entanglement localisation via classical mediator and provided its detailed calculations. Next we looked into the system of two electron spins confined in separate quantum dots as a possible platform to demonstrate the localisation. We computed entanglement between the spins in a toy model where only one electronic spin is coupled to a single-qubit environment. In the limit of pure dephasing we found that starting with a suitable initial state, entanglement can indeed be localised via interactions with classical local environment. In the future it is necessary to extend this model to more spins in the environment and to more natural initial states.

8 Appendix

Refer to above section 4.1, this is how the trace $\text{tr}(\rho^2)$ can be calculated:
If the quantum system is in pure state, given that $\rho = |\psi\rangle\langle\psi|$,

$$\text{tr}(\rho^2) = \text{tr}(|\psi\rangle\langle\psi|\psi\rangle\langle\psi|). \tag{56}$$

In addition, we know the fact that $\text{tr}(|\phi\rangle\langle\psi|) = \langle\psi|\phi\rangle$ and by normalization, $\langle\psi|\psi\rangle = 1$.
Using these, we get

$$\begin{aligned} \text{tr}(\rho^2) &= \langle\psi|\psi\rangle\langle\psi|\psi\rangle \\ &= (1)(1) \\ &= 1. \end{aligned}$$

If the quantum system is in mixed state, given that $\rho = \sum_j p_j|\psi_j\rangle\langle\psi_j|$,

$$\begin{aligned} \text{tr}(\rho^2) &= \text{tr}(\sum_i p_i|\psi_i\rangle\langle\psi_i| \sum_j p_j|\psi_j\rangle\langle\psi_j|) \\ &= \text{tr}(\sum_i \sum_j p_i p_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|) \\ &= \sum_{ij} p_i p_j \text{tr}(|\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|) \end{aligned}$$

Similarly, using the fact that $\text{tr}(|\phi\rangle\langle\psi|) = \langle\psi|\phi\rangle$, therefore we get

$$\text{tr}(\rho^2) = \sum_{ij} p_i p_j \mid \langle\psi_j|\psi_i\rangle \mid^2 . \tag{57}$$

Now, we can consider two cases:
For $i = j$, and using the fact that $\langle\psi_i|\psi_i\rangle = 1$, we get

$$\mid \langle\psi_i|\psi_i\rangle \mid^2 = 1. \tag{58}$$

For $i \neq j$, and using the fact that $\langle\psi_i|\psi_j\rangle = 0$, we get

$$\mid \langle\psi_j|\psi_i\rangle \mid^2 = 0. \tag{59}$$

Hence, taking both cases above into account,

$$\begin{aligned} \text{tr}(\rho^2) &= \sum_{ij} p_i p_j \mid \langle\psi_j|\psi_i\rangle \mid^2 \\ &= \sum_i p_i^2 \\ &< 1. \end{aligned}$$

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