Experimentally friendly geometrical criteria for entanglement

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A simple geometrical criterion gives experimentally friendly sufficient conditions for entanglement. Its generalization gives a necessary and sufficient condition. It is linked with a family of entanglement identifiers, which is strictly richer than the family of entanglement witnesses.

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Entanglement is one of the basic features of quantum physics and it is a resource for quantum information science [1]. Thus, detection of entanglement belongs to the mainstream of this field [2]. Today, the most widely used and experimentally feasible detectors of this resource are entanglement witnesses [3]. They are linked with positive but not completely positive maps [4], which are the most universal entanglement identifiers.

We present an alternative approach to entanglement detection. It is rooted in an elementary geometrical fact: if a scalar product of two real vectors $\vec{s}$ and $\vec{e}$ satisfies $\vec{s}\cdot\vec{e} < \vec{e}\cdot\vec{e}$, then $\vec{s} \neq \vec{e}$. This fact was used in, e.g., [5] to derive a powerful series of Bell inequalities, and in [6] it led to sufficient condition for entanglement. Here, it inspires a new family of entanglement identifiers, which are naturally expressed in terms of the correlation functions [7], easily determined by local measurements. This makes them friendly to experiments. The family of our identifiers is richer than the family of the entanglement witnesses and leads to a necessary and sufficient criterion for entanglement.

The bulk of our *presentation* uses systems of many spin-$\frac{1}{2}$ particles (qubits), but the method is applicable to composite systems of arbitrary dimensions. For that in our formulae, one needs to substitute Pauli operators by their Gell-Mann-type generalizations. This allows a complete separability analysis of a multi-partite state, and will be illustrated by an example. Even if the underlying system consists of many qubits, analysis of the so-called k-separability ($k < N$) requires identification of entanglement in the system partitioned into k parts only [8]. Clearly, at least one part will contain two or more qubits, and can be considered as a multi-level system.

An N-qubit density matrix can be put as follows:

$$\rho = \frac{1}{2^N} \sum_{\mu_1,\ldots,\mu_N=0}^{3} T_{\mu_1\ldots\mu_N} \sigma_{\mu_1} \otimes \ldots \otimes \sigma_{\mu_N}, \tag{1}$$

where $\sigma_{\mu_n} \in \{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ is the μ_n'th local Pauli operator of the nth party ($\sigma_0 = \mathbb{1}$) and $T_{\mu_1\ldots\mu_N} \in [-1,1]$ are the components of the (real) extended correlation tensor $\hat{T}$. They are the expectation values $T_{\mu_1\ldots\mu_N} = \mathrm{Tr}[\rho(\sigma_{\mu_1}\otimes\ldots\otimes\sigma_{\mu_N})]$. The extended tensors are elements of a real vector space with the scalar product given by

$$(\hat{X},\hat{Y}) = \sum_{\mu_1,\ldots,\mu_N=0}^{3} X_{\mu_1\ldots\mu_N} Y_{\mu_1\ldots\mu_N}. \tag{2}$$

A state ρ is separable if it can be put as a convex combination of product states, i.e.,

$$\rho_{\mathrm{sep}} = \sum_i p_i \rho_i^{(1)} \otimes \ldots \otimes \rho_i^{(N)}, \tag{3}$$

with $p_i \geq 0$ for all i, and $\sum_i p_i = 1$. Such a state is specified by a separable extended tensor $\hat{T}^{\mathrm{sep}} = \sum_i p_i \hat{T}_i^{\mathrm{prod}}$, where $\hat{T}_i^{\mathrm{prod}} = \hat{T}_i^{(1)} \otimes \ldots \otimes \hat{T}_i^{(N)}$ and each $\hat{T}_i^{(k)}$ describes one qubit state. Thus, definition (3) implies the following simple criterion. If state ρ, endowed with an extended correlation tensor $\hat{T}$, is separable, then there is an (extended) correlation tensor of a pure product state, $\hat{T}^{\mathrm{prod}}$, such that $(\hat{T},\hat{T}^{\mathrm{prod}}) \geq (\hat{T},\hat{T})$. Indeed, assume that $(\hat{T},\hat{T}^{\mathrm{prod}}) < (\hat{T},\hat{T})$ for all product states. Due to separability of $\hat{T}$, it implies that

$$(\hat{T},\hat{T}) = \sum_i p_i (\hat{T},\hat{T}_i^{\mathrm{prod}}) < \sum_i p_i (\hat{T},\hat{T}) = (\hat{T},\hat{T}),$$

which is a contradiction. In other words,

$$\text{if} \quad \max_{\hat{T}^{\mathrm{prod}}} (\hat{T},\hat{T}^{\mathrm{prod}}) < (\hat{T},\hat{T}), \tag{4}$$

then ρ is entangled.

A simple entanglement identifier is obtained if in the space of correlation tensors one introduces an improper scalar product with the summation indices μ_n in (2) running through the values $j_n = 1, 2, 3$ only (sometimes referred to as x, y, z), i.e.,

$$(\hat{X}_N,\hat{Y}_N) = \sum_{j_1,\ldots,j_N=1}^{3} X_{j_1\ldots j_N} Y_{j_1\ldots j_N}. \tag{5}$$

The maximum of the left-hand side of (4) is now given by the highest generalized Schmidt coefficient [9] of tensor $\hat{T}_N$, denoted here as $T_N^{\max}$. This is the maximal value of the N-qubit correlation function, $T_N^{\max} = \max_{\vec{m}_1 \otimes \ldots \otimes \vec{m}_N}(\hat{T}, \vec{m}_1 \otimes \ldots \otimes \vec{m}_N)$ where $\vec{m}_n = (T_x^{(n)}, T_y^{(n)}, T_z^{(n)})$ is a three-dimensional unit vector describing a pure state of the nth party. Therefore,

$$\mathcal{E} = \frac{||\hat{T}_N||^2}{T_N^{\max}}, \tag{6}$$

with $||\hat{T}_N||^2 = (\hat{T}_N, \hat{T}_N)$, is a simple entanglement identifier. If $\mathcal{E} > 1$, the state is non-separable.

A similar result is obtained when the summation indices are restricted to $j_n = 1, 2$. In this case, however, $T_N^{\max}$ refers to a maximization restricted to two-dimensional sections of the correlation tensor.

If one finds that $||\hat{T}_N||^2 > 1$, one can immediately conclude that the measured state is entangled because $T_N^{\max} \leq 1$. Entanglement identification is then reduced to measurements of orthogonal components $T_{j_1 \ldots j_N}$ of the correlation tensor and summing up their squares until $\sum_{j_1,\ldots,j_N} T^2_{j_1 \ldots j_N}$ exceeds unity. In some cases few measurements may suffice. Take for example the Greenberger-Horne-Zeilinger (GHZ) state [10]:

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}\left(|z+\rangle_1 \ldots |z+\rangle_N + |z-\rangle_1 \ldots |z-\rangle_N\right), \tag{7}$$

where $|z\pm\rangle$ are the eigenstates of the σ_z operator. For indices x or y, this state has 2^{N-1} components of the correlation tensor equal to ± 1. Measurement of any *two* of them is sufficient to detect entanglement. Likewise, two measurements suffice to detect entanglement in any of the graph states [11] definable by N-qubit correlations.

Although very simple, the entanglement identifier of Eq. (6) is optimal for some cases. Consider Werner states, e.g. a mixtures of a singlet $|\psi^-\rangle$ with white noise, with the respective weights p and $1-p$. The extended correlation tensor of this state is diagonal, with the entries $(1, -p, -p, -p)$. Thus, $T_2^{\max} = p$, while one has $||\hat{T}_2||^2 = 3p^2$. Thus, $\mathcal{E} > 1$ for all entangled states of the family, i.e. for $p > \frac{1}{3}$. The new tool identifies entanglement also if the singlet state is replaced by any other maximally entangled state. This distinguishes our identifiers from linear witnesses. There is no single linear witness, which detects entanglement of all Bell states.

A higher dimensional example: qutrits. An arbitrary state of a single qutrit can be parameterized by an 8-dimensional real vector, whose components are the mean values of Gell-Mann operators. Pure states correspond to normalized vectors $\vec{n}$ (we use the same normalization factors as in Ref. [13]). The admissible vectors $\vec{n}$ obey an additional condition, see e.g. [13]. A state of two qutrits can be expressed in the operator basis made of tensor products of Gell-Mann operators (including the unit operators). Consider a mixture of maximally entangled state of two qutrits $|\Psi\rangle = \frac{1}{\sqrt{3}}(|11\rangle + |22\rangle + |33\rangle)$ and white noise, with respective weights p and $1-p$. The (not extended) correlation tensor of this state is diagonal: $\hat{T}_2 = \frac{p}{2}\mathrm{diag}[1, -1, 1, 1-1, 1-1, 1]$ and $||\hat{T}_2||^2 = 2p^2$. The maximization in our criterion is over all normalized vectors $\vec{n}_A$ and $\vec{n}_B$. It gives $\max_{\vec{n}_A, \vec{n}_B}(\hat{T}_2, \vec{n}_A \otimes \vec{n}_B) = \frac{p}{2}$. This cannot be smaller than the maximum over the admissible $\vec{n}$'s only. Thus, the state is entangled for $p > \frac{1}{4}$. The same value is reported in, e.g., Ref. [13]. The fact that this can be obtained ignoring the condition for admissible $\vec{n}$'s is a surprising bonus.

Generalized Werner states of N qubits. Consider mixtures of a GHZ state with the white noise:

$$\rho(p) = p|\mathrm{GHZ}_N\rangle\langle \mathrm{GHZ}_N| + (1-p)\frac{1}{2^N}\mathbb{1}. \tag{8}$$

Since white noise exhibits no correlations, the components $T_{j_1 \ldots j_N}$ of the correlation tensor of $\rho(p)$ are related to the components $T^{\mathrm{GHZ}}_{j_1 \ldots j_N}$ of the GHZ state by $T_{j_1 \ldots j_N} = p\, T^{\mathrm{GHZ}}_{j_1 \ldots j_N}$. Again, $T^{\max} = p$. Applying condition (6) with the sums over $j_n = 1, 2, 3$ one finds that $\mathcal{E} > 1$ for the admixture parameter $p > 1/(2^{N-1} + \frac{1+(-1)^N}{2})$. The same critical value for N even was found by Pittenger and Rubin [12], who used the PPT criterion. However, for N odd their result is still $p > 1/(2^{N-1} + 1)$ whereas in our case the term $\frac{1+(-1)^N}{2}$ vanishes and our criterion in its simplest form is weaker than PPT.

The reason why condition (6) is not as efficient as PPT is that for odd N the GHZ states have no $T_{z \ldots z}$ correlations. Nevertheless, they have additional correlations between local z measurements on even numbers of particles. These correlations are described by their corresponding $2^{N-1} - 1$ components of the extended correlation tensor. Our simplest criterion does not utilize these correlations. If one attempts to include them by using condition (4) with indices $\mu_n = 0, 1, 2, 3$ the situation gets even worse.

Generalized scalar product. The last example indicates that taking into account more correlations in the criterion does not guarantee better entanglement detection. For a success, one needs a proper combination of the correlations. To identify it, one may consider generalized scalar products, defined via a positive semi-definite metric G:

$$(\hat{X}, \hat{Y})_G = \sum_{\substack{\mu_1,\ldots,\mu_N, \\ \nu_1,\ldots,\nu_N = 0}}^{3} X_{\mu_1 \ldots \mu_N} G_{\mu_1 \ldots \mu_N, \nu_1 \ldots \nu_N} Y_{\nu_1 \ldots \nu_N}. \tag{9}$$

If one can find a metric for which

$$\max_{\hat{T}^{\mathrm{prod}}} (\hat{T}, \hat{T}^{\mathrm{prod}})_G < (\hat{T}, \hat{T})_G, \tag{10}$$

then the state ρ described by its (extended) correlation tensor $\hat{T}$ is entangled.

This criterion is very powerful and often it is easy to apply: a suitable metric can be guessed from the structure of the correlation tensor of the state in question.

Later on we will prove that criterion (10) is also necessary for a state to be entangled.

Generalized Werner states for odd N. To illustrate criterion (10), let us return to the generalized Werner states for an odd number of qubits. Consider a diagonal metric $G_{\mu_1...\mu_N,\nu_1...\nu_N} = G_{\mu_1...\mu_N}\delta_{\mu_1...\mu_N,\nu_1...\nu_N}$, with $G_{0...0} = 0$, all $G_{j_1...j_N} = 1$, all $G_{\mu_1...\mu_N}$ with at least one $\mu_n = 0$ equal to $\omega = 1/(2^{N-1}-1)$. This makes the left-hand side of condition (10) equal to $p(2^{N-1}-1)\omega = p$. This is seen directy once one writes down the 'spatial' components of vectors defining single qubit states, $\hat{T}^{(n)}$, $n = 1, ..., N$, in the spherical coordinates. The optimal choice is to put all the local vectors along z directions. The right-hand side of (10) is given by $p^2(2^{N-1}+(2^{N-1}-1)\omega) = p^2(2^{N-1}+1)$. Thus, the condition reveals entanglement of the generalized Werner states for $p > 1/(2^{N-1}+1)$, exactly as given by the PPT criterion.

Colored noise. Consider, e.g., a two-qubit state

$$\rho_C(p) = p|\psi^-\rangle\langle\psi^-| + (1-p)|z+\rangle\langle z+|\otimes|z+\rangle\langle z+|. \quad (11)$$

Its correlation tensor has six non-vanishing elements: $T_{00} = 1$, $T_{xx} = T_{yy} = -p$, $T_{zz} = 1-2p$, and $T_{z0} = T_{0z} = 1-p$. The whole range of p, for which $\rho_C(p)$ is entangled, can be found using a diagonal metric with the following non-zero elements: $G_{11} = G_{22} = 1$ and $G_{03} = G_{30} = p$. To find maximum of (10), we again use the spherical coordinates for the spatial elements of the vectors describing the single qubit pure states: $T_x^{(n)} = \sin\theta_n\cos\varphi_n$, $T_y^{(n)} = \sin\theta_n\sin\varphi_n$, and $T_z^{(n)} = \cos\theta_n$. This gives for the left-hand side $L = \max_{\theta_1,\theta_2,\varphi_1,\varphi_2}[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2))p - (\cos\theta_1+\cos\theta_2)p^2]$. Only the first term depends on φ's and it is maximal for $\cos(\varphi_1-\varphi_2) = \pm 1$. We put $\cos(\varphi_1-\varphi_2) = -1$ and prove that L is maximal for $\theta_1 = \theta_2 \equiv \theta$ such that $\cos\theta = 1-p$ (the choice of $\cos(\varphi_1-\varphi_2) = +1$ gives the same maxima). For this choice $\frac{\partial L}{\partial\theta_1} = \frac{\partial L}{\partial\theta_2} = 0$, i.e. the allowed θ's define stationary points. To show that they correspond to a maximum we compute second derivatives $\frac{\partial^2 L}{\partial\theta_1^2} = \frac{\partial^2 L}{\partial\theta_2^2} = -p$ and $\frac{\partial^2 L}{\partial\theta_2\partial\theta_1} = p(1-p)^2$ at the stationary points. The derivatives $\frac{\partial^2 L}{\partial\theta_n^2}$ are negative for all p, as it should be for a maximum. It remains to show that the Hessian determinant is always positive. In our case it is given by $-p^6+4p^5-6p^4+4p^3$ and indeed it is always strictly positive in the allowed range of p. Using the optimal θ's the left-hand side equals $L = 2p-2p^2+p^3$. The right-hand side of condition (10) is given by $R = 2p-2p^2+2p^3$ and it is bigger than L for all p. Thus, the state (11) is entangled for any, no matter how small, admixture of $|\psi^-\rangle$.

Condition for density operators. All the steps in the proof of condition (10) can be done without any reference to a specific representation of the state. As a generalized scalar product in the operator space one just has to take a weighted-trace with the positive semi-definite superoperator G, i.e., $(\rho_1,\rho_2)_G = \mathrm{Tr}(\rho_1 G\rho_2)$. The sufficient condition for entanglement now reads: if there is a positive superoperator G, such that

$$\max_{\rho_{\mathrm{prod}}} \mathrm{Tr}(\rho G\rho_{\mathrm{prod}}) < \mathrm{Tr}(\rho G\rho), \quad (12)$$

where we maximize over all pure product states ρ_{prod}, then state ρ is entangled.

Bound entanglement. To show usefulness of condition (12), consider a state of two four-level systems which is bound entangled:

$$\rho_{\mathrm{B}} = \frac{\rho_0 + p\rho_+ + q\rho_-}{1+p+q}. \quad (13)$$

The contributions to ρ_{B} represent one separable and two entangled states. The separable contribution is

$$\rho_0 = \frac{1}{4}\left(|++\rangle\langle++| + |--\rangle\langle--| + |2\ 2\rangle\langle 2\ 2| + |3\ 3\rangle\langle 3\ 3|\right) \quad (14)$$

with $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The two entangled contributions are

$$\rho_\pm = \frac{1}{4}\left(|\phi_{02}^\pm\rangle\langle\phi_{02}^\pm| + |\phi_{12}^\pm\rangle\langle\phi_{12}^\pm| + |\phi_{03}^\pm\rangle\langle\phi_{03}^\pm| + |\phi_{13}^\pm\rangle\langle\phi_{13}^\pm|\right), \quad (15)$$

where for $j = 0, 1$ we define $|\phi_{j2}^\pm\rangle = \frac{1}{\sqrt{2}}(|j\ 2\rangle \pm |2\ j\rangle)$ and $|\phi_{j3}^\pm\rangle = \frac{1}{\sqrt{2}}(|j\ 3\rangle \pm |3\ j'\rangle)$ with $j' = (j+1) \mod 2$. For convenience, we introduce a real parameter b defined by $p = \frac{b+\sqrt{2}}{2}$ and $q = \frac{b-\sqrt{2}}{2}$ and consider $b \geq \sqrt{2}$. The state ρ_{B} has a positive partial transposition for all $b \geq \sqrt{2}$. In fact, it represents a partial transposition of the state generating a classical probability distribution described by Renner and Wolf in their search for bound information [14]. To reveal entanglement of ρ_{B}, one can use our criterion (12) with G representing projection on ρ_+, $G = |\rho_+)(\rho_+|$, where $|.)(.|$ denotes a projector in the Hilbert-Schmidt space. Since ρ_+ is orthogonal to both ρ_0 and ρ_- and $\mathrm{Tr}\rho_+^2 = \frac{1}{4}$ the right-hand side reads $(\rho_{\mathrm{B}},\rho_{\mathrm{B}})_G = (\rho_{\mathrm{B}},\rho_+)(\rho_+,\rho_{\mathrm{B}}) = \frac{p^2}{16(1+b)^2}$. The left-hand side is given by $\max(\rho_{\mathrm{B}},\rho_{\mathrm{prod}})_G = \max(\rho_{\mathrm{B}},\rho_+)(\rho_+,\rho_{\mathrm{prod}}) = \frac{p}{32(1+b)}$, where $\max(\rho_+,\rho_{\mathrm{prod}}) = \frac{1}{8}$. A product state for which this maximum is achieved can be chosen among the products which enter the definitions of $|\phi^\pm\rangle$'s. Numerical results agree with $\frac{1}{8}$ being the maximum over the choice of any product state. Thus, we identify entanglement whenever $\frac{b+\sqrt{2}}{b+1} > 1$, i.e. for all $b \geq \sqrt{2}$.

Necessity of the criterion. The examples suggest that the conditions are *necessary* for entanglement, i.e.

- for every entangled state there exists a metric G such that inequalities (12) or (10) hold.

Proof. The separable states ρ_{sep} form a convex and compact set, $\mathcal{S}$, in the space of Hermitian operators, with a standard scalar product $(\varrho|\varrho') = \mathrm{Tr}(\varrho\varrho')$. Any entangled state ρ_{ent} is at a non-zero distance from $\mathcal{S}$. Consider two separable states ρ_0 and ρ_1, and let ρ_0 be the separable state, which minimizes the distance to $\rho_{\rm ent}$. Due to convexity of $\mathcal{S}$, any convex combination of these states, $\rho_\lambda = (1-\lambda)\rho_0 + \lambda\rho_1$ is separable too. Thus, the norm (length) of operator $\gamma = \rho_{\rm ent} - \rho_\lambda$ is strictly positive, and not smaller than the norm of $\gamma_0 = \rho_{\rm ent} - \rho_0$:

$$\|\gamma\|^2 = \lambda^2\|\rho_0 - \rho_1\|^2 + 2\lambda(\rho_0 - \rho_1|\gamma_0) + \|\gamma_0\|^2 \geq \|\gamma_0\|^2 . \tag{16}$$

The inequality can be saturated only for $\lambda = 0$, otherwise it contradicts our assumptions. Thus, the derivative of the left-hand side with respect to λ, at $\lambda = 0$, cannot be negative. This requires that $(\rho_0|\gamma_0) \geq (\rho_1|\gamma_0)$ for all separable ρ_1. We also have $\|\gamma_0\|^2 = (\rho_{\rm ent} - \rho_0|\gamma_0) > 0$. Consequently, for all separable states $\rho_{\rm sep}$, one has

$$(\rho_{\rm ent}|\gamma_0) > (\rho_{\rm sep}|\gamma_0). \tag{17}$$

Moreover, the definition of γ_0 implies that $(\rho_{\rm ent}|\gamma_0) > 0$. To show this, we decompose $\rho_{\rm ent}$ and ρ_0 into $\frac{1}{d}\mathbb{1} + X$ and $\frac{1}{d}\mathbb{1} + Y_0$, respectively. Convexity of $\mathcal{S}$, together with the fact that the maximally mixed state $\frac{1}{d}\mathbb{1}$ is separable, require that $\rho_0^\lambda = \frac{1}{d}\mathbb{1} + \lambda Y_0$ is a separable state for all $0 \leq \lambda \leq 1$. In this range the lenght of $\delta = \rho_{\rm ent} - \rho_0^\lambda = X - \lambda Y_0$ reads

$$\|\delta\|^2 = \|X\|^2 - 2\lambda(X|Y_0) + \lambda^2\,\|Y_0\|^2 \geq \|\gamma_0\|^2 > 0 \tag{18}$$

For (18) to hold, one needs $\frac{d}{d\lambda}(\|\delta\|^2) \leq 0$ at $\lambda = 1$. This requires that $(X|Y_0) \geq \|Y_0\|^2$. Since $X - Y_0 = \gamma_0$ is a traceless operator one has $(X - Y_0|Y_0) = (\gamma_0|\rho_0)$. Thus the last inequality is equivalent to $(\gamma_0|\rho_0) \geq 0$. By combining this fact with inequality (17) and the separability of ρ_0, we arrive at the strict positivity of $(\rho_{\rm ent}|\gamma_0)$. This allows us to multiply both sides of (17) by $(\rho_{\rm ent}|\gamma_0)$ without reversing the inequality's direction. It results in $(\rho_{\rm ent}|\gamma_0)(\gamma_0|\rho_{\rm ent}) > (\rho_{\rm sep}|\gamma_0)(\gamma_0|\rho_{\rm ent})$ and we conclude that, for every entangled state $\rho_{\rm ent}$ one can find a positive semi-definite superoperator G_{γ_0}, the action of which can be symbolically expressed as $G_{\gamma_0} = |\gamma_0\,)(\,\gamma_0|$, such that for all separable states $\rho_{\rm sep}$

$$(\rho_{\rm sep}|\; G_{\gamma_0}\; |\rho_{\rm ent}) < (\rho_{\rm ent}|\; G_{\gamma_0}\; |\rho_{\rm ent}). \tag{19}$$

Since this inequality is valid for all separable states $\rho_{\rm sep}$, it is also valid for all pure product states $\rho_{\rm prod}$. QED.

Relation with the entanglement witnesses. The Hermitian operator γ_0 of the above proof is related to the entanglement witness W identifying entanglement of $\rho_{\rm ent}$. Simply, one has $W = q\mathbb{1} - \gamma_0$, where $q = \max_{\rho_{\rm sep}\in\mathcal{S}}(\rho_{\rm sep}|\gamma_0)$. Indeed, $(W|\rho_{\rm sep}) \geq 0$ for all separable states $\rho_{\rm sep}$ and $(W|\rho_{\rm ent}) < 0$. Conversely, each entanglement witness W defines a hermitian operator $\gamma_0 = w\mathbb{1} - W$, where $w = \max_{\rho_{\rm sep}\in\mathcal{S}}(W|\rho_{\rm sep})$. However, one should stress that many entanglement identifiers G *do not* have their witness counterparts. If G is not a projector, there is no entanglement witness corresponding to it. In particular, *only one* of the identifiers of our examples can be associated with an entanglement witness!

Summary. We have derived simple sufficient conditions for entanglement of distributed quantum states. Their generalization gives a necessary and sufficient separability criterion. The set of entanglement identifiers defined by our criterion is strictly richer than the set of entanglement witnesses. Moreover, the discussed examples indicate that many identifiers not corresponding to any standard entanglement witness are particularly useful.

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