

Experimentally accessible geometrical separability criteria

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Abstract

We present an intuitive geometrical approach to entanglement detection. It allows one to formulate simple and experimentally feasible sufficient conditions for entanglement. Within the approach we derive the necessary and sufficient condition for separability and discuss its relation with entanglement witnesses and positive maps.

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1. Introduction

Entanglement is one of the most fundamental features of quantum physics and it is recognized as the key resource in quantum information processing [1]. Thus, its detection attracts much research interest [2]. In this respect, the experiment yields best to the methods based on entanglement witnesses [3]⁵ derived from positive, but not completely positive, maps [4].

We present an alternative, purely geometrical approach to entanglement identification. It leads to a sufficient criterion for entanglement expressed in terms of simple conditions on the correlation functions⁶, easily tested by local measurements. A generalization of the simple criterion gives a necessary and sufficient test for entanglement applicable to composite systems of arbitrary dimensions. There it allows for analysis of the so-called k -separability [6] and, consequently, a complete separability analysis of distributed states. A somewhat more extended version of our results was published in [7].

2. Simple criteria for many qubits

Consider a system composed of many spin- $\frac{1}{2}$ particles (qubits). A useful representation of an arbitrary state of many

qubits is given by the correlation tensor

$$\rho = \frac{1}{2^N} \sum_{\mu_1, \dots, \mu_N=0}^3 T_{\mu_1 \dots \mu_N} \sigma_{\mu_1} \otimes \dots \otimes \sigma_{\mu_N}, \quad (1)$$

where $\sigma_{\mu_n} \in \{I, \sigma_x, \sigma_y, \sigma_z\}$ is the μ_n th local Pauli operator of the n th party and $T_{\mu_1 \dots \mu_N} = \text{Tr}[\rho(\sigma_{\mu_1} \otimes \dots \otimes \sigma_{\mu_N})]$ are the components of the generalized correlation tensor \hat{T} . They are accessible to standard experiments performed for e.g. quantum state estimation (tomography). We phrase our simple criteria in terms of the correlation tensor.

A state ρ is separable (not entangled) if it can be put as a convex combination of product states, i.e.

$$\rho_{\text{sep}} = \sum_i p_i \rho_i^{(1)} \otimes \dots \otimes \rho_i^{(N)}, \quad (2)$$

with $p_i \geq 0$ for all i , and $\sum_i p_i = 1$. In the language of the correlation tensors this decomposition reads $\hat{T}_i^{\text{sep}} = \sum_i p_i \hat{T}_i^{\text{(prod)}}$, where $\hat{T}_i^{\text{(prod)}} = \hat{T}_i^{(1)} \otimes \dots \otimes \hat{T}_i^{(N)}$ and each $\hat{T}_i^{(k)}$ describes a one-qubit state. The correlation tensors form a real vector space with a usual scalar product

$$\langle \hat{X}, \hat{Y} \rangle = \sum_{\mu_1, \dots, \mu_N} X_{\mu_1 \dots \mu_N} Y_{\mu_1 \dots \mu_N}. \quad (3)$$

For our purpose, we allow generalized products by permitting the summation to run over selected subsets of the whole range

⁵ For the PPT criterion, see Peres [3].

⁶ A set of different sufficient criteria based on correlation functions can be found in e.g. [5].

of the indices $\mu_n = 0, 1, 2, 3$, e.g. $\mu_n = 1, 2$ (in which cases we shall call the indices j_n). The separable correlation tensors form a convex set in this space. Consequently, when \hat{T} is the tensor of ρ , one has the implication

$$\rho \text{ is separable} \Rightarrow \exists_{\hat{T}^{\text{prod}}} (\hat{T}, \hat{T}^{\text{prod}}) \geq (\hat{T}, \hat{T}), \quad (4)$$

or, equivalently:

$$\max_{\hat{T}^{\text{prod}}} (\hat{T}, \hat{T}^{\text{prod}}) < (\hat{T}, \hat{T}) \Rightarrow \rho \text{ is entangled.} \quad (5)$$

To see that (4) is true, assume that $(\hat{T}, \hat{T}^{\text{prod}})$ is strictly smaller than (\hat{T}, \hat{T}) for all product states and \hat{T} is separable. Then we have $(\hat{T}, \hat{T}) = \sum_i p_i (\hat{T}, \hat{T}^{\text{prod}})$. By assumption it is strictly smaller than $\sum_i p_i (\hat{T}, \hat{T}) = (\hat{T}, \hat{T})$, which is a contradiction.

Implication (5) leads to a simple and useful criterion when the summation in the scalar product is restricted to the indices j_n with values $j_n = 1, 2$ or $j_n = 1, 2, 3$. In this case, the maximization on the left-hand side of (5) is given by the highest Schmidt coefficient, T^{max} , of tensor \hat{T} [8]. Therefore, the quantity

$$\mathcal{E} = \frac{\|\hat{T}\|^2}{T^{\text{max}}}, \quad (6)$$

where $\|\hat{T}\|^2 = (\hat{T}, \hat{T})$, is a simple entanglement witness. If $\mathcal{E} > 1$ there exists at least bipartite entanglement in the N -qubit state. Moreover, since $T^{\text{max}} \leq 1$ then the state is entangled if $\|\hat{T}\|^2 > 1$. To establish the latter may sometimes require a very limited number of measurements. For example, in the case of the Greenberger–Horne–Zeilinger (GHZ) state [9], measurement of *two* correlations is sufficient to detect entanglement (for indices x or y , this state has 2^{N-1} components of the correlation tensor equal to ± 1). Likewise, two measurements suffice to detect entanglement in any of the graph states [10].

Moreover, condition (6) is quite universal. It can show entanglement of all Bell states ($\mathcal{E} = 3$) in the same setup even if there is no single linear witness, which detects entanglement of all these states.

Finally, our method is applicable to composite systems of arbitrary dimensions (higher than qubits). For that in our formulae, one needs to replace Pauli operators by their Gell–Mann-type generalizations.

3. Generalized scalar product

Despite its advantages, condition (6) has some shortcomings. When applied to detect entanglement in mixtures of a GHZ state with the white noise:

$$\rho(p) = p|\text{GHZ}_N\rangle\langle\text{GHZ}_N| + (1-p)\frac{1}{2^N}I, \quad (7)$$

it detects all the entanglement detectable by the PPT criterion (cf [11]) in states of even numbers of qubits. For an odd number of qubits, however, condition (6) with the sums over $j_n = 1, 2, 3$ gives a weaker entanglement criterion than PPT [7]. Moreover, inclusion of additional correlations in the scalar product could not cure the weakness.

The last example indicates that for success, one may need a proper combination of the available correlations.

To identify it, one may consider generalized scalar products, defined via a positive semi-definite metric G :

$$(\hat{X}, \hat{Y})_G = \sum_{\mu_1, \dots, \mu_N; \nu_1, \dots, \nu_N} X_{\mu_1 \dots \mu_N} G_{\mu_1 \dots \mu_N, \nu_1 \dots \nu_N} Y_{\nu_1 \dots \nu_N}. \quad (8)$$

If one can find a metric for which

$$\max_{\hat{T}^{\text{prod}}} (\hat{T}, \hat{T}^{\text{prod}})_G < (\hat{T}, \hat{T})_G \quad (9)$$

then the state ρ described by its (extended) correlation tensor \hat{T} is entangled.

To illustrate criterion (9), let us return to the odd particle generalized Werner states. Consider a diagonal metric $G_{\mu_1 \dots \mu_N, \nu_1 \dots \nu_N} = G_{\mu_1 \dots \mu_N} \delta_{\mu_1 \dots \mu_N, \nu_1 \dots \nu_N}$ which couples all the z -correlations: $G_{zz0 \dots 0} = \dots = G_{0 \dots 0zz} = G_{zz0 \dots 0} = \dots = 1/(2^{N-1} - 1) \equiv \omega$. The left-hand side of condition (9) equals $p(2^{N-1} - 1)\omega = p$. The optimal choice of local tensors is $\hat{T}_{j_n}^{(n)} = (1, 0, 0, 1)$ for all $n = 1, \dots, N$. The right-hand side of (9) is given by $p^2(2^{N-1} + (2^{N-1} - 1)\omega) = p^2(2^{N-1} + 1)$. Thus, the condition reveals entanglement of the generalized Werner states for $p > 1/(2^{N-1} + 1)$, exactly as given by the PPT criterion.

4. Condition for density operators

All the steps in the proof of condition (9) can be done without any reference to a specific representation of the state. As a generalized scalar product in the operator space one just has to take a weighted trace with the positive semi-definite superoperator G , i.e. $(\rho_1, \rho_2)_G = \text{Tr}(\rho_1 G \rho_2)$. The sufficient condition for entanglement now reads: if there is a positive superoperator G , such that

$$\max_{\rho_{\text{prod}}} \text{Tr}(\rho G \rho_{\text{prod}}) < \text{Tr}(\rho G \rho), \quad (10)$$

where we maximize over all pure product states ρ_{prod} , then state ρ is entangled.

Condition (10) was used to identify bound entanglement, which is especially difficult to detect [7].

5. Necessary and sufficient conditions and relation to entanglement witnesses

In [7], we showed that to every entanglement witness W there corresponds a superoperator $G(W)$, which identifies entanglement in all the states where W does. In the Hilbert–Schmidt space of operators, the superoperator $G(W)$ is a one-dimensional projector on a (super) vector $\gamma_0 = wI - W$, where $w = \max_{\rho_{\text{sep}} \in \mathcal{S}} (W, \rho_{\text{sep}})$. A simple proof of this statement may go as follows. Notice that $\text{Tr}(\rho G(W) \rho') = \text{Tr}(\rho \gamma_0) \text{Tr}(\gamma_0 \rho')$ and that for a state ρ with entanglement identifiable by W one has $\text{Tr}(\rho \gamma_0) > w \geq 0$. Thus, if

$$\text{Tr}(\gamma_0 \rho) > \text{Tr}(\gamma_0 \rho_{\text{sep}}) \quad (11)$$

for all separable ρ_{sep} , then condition (10) with $G = G(W)$ identifies entanglement of ρ . When W identifies entanglement of ρ then $\text{Tr}(\gamma_0 \rho) > w$ and $\text{Tr}(\gamma_0 \rho_{\text{sep}}) \leq w$, i.e. condition (11) is satisfied and, consequently, $G(W)$ identifies entanglement of ρ too. One should stress, however, that many entanglement

identifiers G do not have their witness counterparts. In particular, when G is not a one-dimensional projector, there is no entanglement witness corresponding to it. We found these superoperators particularly interesting entanglement identifiers.

6. Conclusions

We have derived handy sufficient conditions for the entanglement of distributed quantum states. It works in the Hilbert spaces of arbitrary dimension and can be applied for arbitrarily selected subsystems (k -separability problem). Their generalization gives a necessary and sufficient separability criterion. The set of entanglement identifiers defined by our criterion is strictly richer than the set of entanglement witnesses.

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