Bell inequality with an arbitrary number of settings and its applications

Koji Nagata,[1,2] Wiesław Laskowski,[1] and Tomasz Paterek[1,3]

[1]Instytut Fizyki Teoretycznej i Astrofizyki, Uniwersytet Gdański, PL-80-952 Gdańsk, Poland
[2]National Institute of Information and Communications Technology, 4-2-1 Nukuikita, Koganei, Tokyo 184-8795, Japan
[3]The Erwin Schrödinger International Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Vienna, Austria

(Dated: February 1, 2008)

Based on a geometrical argument introduced by Żukowski, a new multisetting Bell inequality is derived, for the scenario in which many parties make measurements on two-level systems. This generalizes and unifies some previous results. Moreover, a necessary and sufficient condition for the violation of this inequality is presented. It turns out that the class of non-separable states which do not admit local realistic description is extended when compared to the two-setting inequalities. However, supporting the conjecture of Peres, quantum states with positive partial transposes with respect to all subsystems do not violate the inequality. Additionally, we follow a general link between Bell inequalities and communication complexity problems, and present a quantum protocol linked with the inequality, which outperforms the best classical protocol.

PACS numbers: 03.65.Ud, 03.65.Ca, 03.67.Mn

I. INTRODUCTION

Any theory based on classical concepts, such as locality and realism, predicts bounds on the correlations between measurement outcomes obtained in space-separation [1]. These bounds are known as Bell inequalities (see [2] for reviews). Profoundly, the correlations measured on certain quantum states violate Bell inequalities, implying incompatibility between the quantum and classical world-views. Which are these non-classical states of quantum mechanics? Here, we present a tool which allows one to extend the class of non-classical states, and gives further evidence that there may exist many-particle entangled states whose correlations admit a local realistic description.

Despite their fundamental role, with the emergence of quantum information [3], Bell inequalities have found practical applications. Quantum advantages of certain protocols, like quantum cryptography [4] or quantum communication complexity [5], are linked with Bell inequalities. Thus, new inequalities lead to new schemes. As an example, we present communication complexity problem associated with the new multisetting inequality.

Specifically, based on a geometrical argument by Żukowski [6], a Bell inequality for many observers, each choosing between arbitrary number of dichotomic observables, is derived. Many previously known inequalities are special cases of this new inequality, e.g. Clauser-Horne-Shimony-Holt inequality [7] or tight two-setting inequalities [8]. The new inequalities are maximally violated by the Greenberger-Horne-Zeilinger (GHZ) states [9]. Many other states violate them, including the states which satisfy two-settings inequalities [10] and bound entangled states [11]. This is shown using the necessary and sufficient condition for the violation of the inequalities. Finally, it is proven that the Bell operator has only two non-vanishing eigenvalues which correspond to the GHZ states, and thus has a very simple form. This form is utilized to show that quantum states with positive partial transposes [12] with respect to all subsystems (in general the necessary but not sufficient condition for entanglement [13]) do not violate the new inequalities. This is further supporting evidence for a conjecture by Peres that positivity of partial transposes could lead us to the existence of a local realistic model [14].

The paper is organized as follows. In section II we present the multisetting inequality. In section III the necessary and sufficient condition for a violation of the inequality is derived, and examples of non-classical states are given. Next, we support the conjecture by Peres in section IV, and follow the link with communication complexity problems in section V. Section VI summarizes this paper.

II. MULTISETTING BELL INEQUALITIES

Consider N separated parties making measurements on two-level systems. Each party can choose one of M dichotomic, of values ± 1, observables. In this scenario parties can measure M^N correlations $E_{m_1 \dots m_N}$, where the index $m_n = 0, \dots, M-1$ denotes the setting of the nth observer. A general Bell expression, which involves these correlations with some coefficients $c_{m_1 \dots m_N}$, can be written as:

$$\sum_{m_1,\dots,m_N=0}^{M-1} c_{m_1 \dots m_N} E_{m_1 \dots m_N} = \vec{C} \cdot \vec{E}. \quad (1)$$

In what follows we assume certain form of coefficients $c_{m_1 \dots m_N}$, and compute local realistic bound as a maximum of a scalar product $|\vec{C} \cdot \vec{E}^{LR}|$. The components of vector $\vec{E}^{LR}$ have the usual form:

$$E^{LR}_{m_1 \dots, m_N} = \int d\lambda \rho(\lambda) I^1_{m_1}(\lambda) \dots I^N_{m_N}(\lambda), \quad (2)$$

where λ denotes a set of hidden variables, $\rho(\lambda)$ their distribution, and $I^n_{m_n}(\lambda)=\pm1$ the predetermined result of nth observer under setting m_n. The quantum prediction for the Bell expression (1) is given by a scalar product of $\vec{C}\cdot\vec{E}^{QM}$. The components of $\vec{E}^{QM}$, according to quantum theory, are given by:

$$E^{QM}_{m_1...m_N}=\mathrm{Tr}\left(\rho\vec{m}_1\cdot\vec{\sigma}^1\otimes...\otimes\vec{m}_N\cdot\vec{\sigma}^N\right), \qquad (3)$$

where ρ is a density operator (general quantum state), $\vec{\sigma}^n=(\sigma^n_x,\sigma^n_y,\sigma^n_z)$ is a vector of local Pauli operators for nth observer, and $\vec{m}_n$ denotes a normalized vector which parameterizes observable m_n for the nth party.

Assume that local settings are parameterized by a single angle: $\phi^n_{m_n}$. In the quantum picture we restrict observable vectors $\vec{m}_n$ to lie in the equatorial plane:

$$\vec{m}_n\cdot\vec{\sigma}^n=\cos\phi^n_{m_n}\sigma^n_x+\sin\phi^n_{m_n}\sigma^n_y. \qquad (4)$$

Take the coefficients $c_{m_1...m_N}$ in a form:

$$c_{m_1...m_N}=\cos(\phi^1_{m_1}+...+\phi^N_{m_N}), \qquad (5)$$

with the angles given by:

$$\phi^n_{m_n}=\frac{\pi}{M}m_n+\frac{\pi}{2MN}\eta. \qquad (6)$$

The number $\eta=1,2$ is fixed for a given experimental situation, i.e. M and N, and equals:

$$\eta=[M+1]_2[N]_2+1, \qquad (7)$$

where $[x]_2$ stands for x modulo 2. The local realistic bound is given by a maximal value of the scalar product $|\vec{C}\cdot\vec{E}^{LR}|$. The maximum is attained for deterministic local realistic models, as they correspond to the extremal points of a correlation polytope. Thus, the following inequality appears:

$$|\vec{C}\cdot\vec{E}^{LR}|\leq \max_{I^1_0,...,I^N_{M-1}=\pm1}\left\{\sum_{m_1,...,m_N=0}^{M-1}\cos(\phi^1_{m_1}+...+\phi^N_{m_N})I^1_{m_1}...I^N_{m_N}\right\}, \qquad (8)$$

where we have shortened the notation $I^n_{m_n}\equiv I^n_{m_n}(\lambda)$. Since $\cos(\phi^1_{m_1}+...+\phi^N_{m_N})=\mathrm{Re}\left(\prod_{n=1}^N\exp(i\phi^n_{m_n})\right)$ and the predetermined results, $I^n_{m_n}=\pm1$, are real, the right-hand side of this inequality can be written as:

$$\sum_{m_1,...,m_N=0}^{M-1}\mathrm{Re}\left(\prod_{n=1}^N\exp(i\phi^n_{m_n})I^n_{m_n}\right). \qquad (9)$$

Moreover, since inequality (8) involves the sum of all possible products of local results respectively multiplied by the cosines of all possible sums of local angles, the right-hand side can be further reduced to involve the product of sums:

$$\mathrm{Re}\left(\prod_{n=1}^N\sum_{m_n=0}^{M-1}\exp(i\phi^n_{m_n})I^n_{m_n}\right). \qquad (10)$$

Inserting the angles (6) into this expression results in:

$$\mathrm{Re}\left(\exp(i\frac{\pi}{2M}\eta)\prod_{n=1}^N\sum_{m_n=0}^{M-1}\exp(i\frac{\pi}{M}m_n)I^n_{m_n}\right), \qquad (11)$$

where the factor $\exp(i\frac{\pi}{2M}\eta)$ comes from the term $\frac{\pi}{2MN}\eta$ in (6), which is the same for all parties.

One can decompose a complex number given by the sum in (11) into its modulus R_n, and phase Φ_n:

$$\sum_{m_n=0}^{M-1}\exp(i\frac{\pi}{M}m_n)I^n_{m_n}=R_ne^{i\Phi_n}. \qquad (12)$$

We maximize the length of this vector on the complex plane. The length of the sum of any two complex numbers $|z_1+z_2|^2$ is given by the law of cosines as $|z_1|^2+|z_2|^2+2|z_1||z_2|\cos\varphi$, where φ is the angle between the corresponding vectors. To maximize the length of the sum one should choose the summands as close as possible to each other. Since in our case all vectors being summed are rotated by multiples of $\frac{\pi}{M}$ from each other, the simplest optimal choice is to put all $I^n_{m_n}=1$. In this case one has:

$$R_n^{\max}=\left|\sum_{m_n=0}^{M-1}\exp(i\frac{\pi}{M}m_n)\right|=\left|\frac{2}{1-\exp(i\frac{\pi}{M})}\right|, \qquad (13)$$

where the last equality follows from the finite sum of numbers in the geometric progression (any term in the sum is given by the preceding term multiplied by $e^{i\pi/M}$). The denominator inside the modulus can be transformed to $\exp(i\frac{\pi}{2M})\left[\exp(-i\frac{\pi}{2M})-\exp(i\frac{\pi}{2M})\right]$, which reduces to $-2i\exp(i\frac{\pi}{2M})\sin\left(\frac{\pi}{2M}\right)$. Finally, the maximal length reads:

$$R_n^{\max}=\frac{1}{\sin\left(\frac{\pi}{2M}\right)}, \qquad (14)$$

where the modulus is no longer needed since the argument of sine is small. Moreover, since the local results for each party can be chosen independently, the maximal length $R_n^{\max}$ does not depend on particular n, i.e. $R_n^{\max}=R^{\max}$.

Since $R^{\max}$ is a positive real number its Nth power can be put to multiply the real part in (11), and one finds $|\vec{C}\cdot\vec{E}^{LR}|$ to be bounded by:

$$|\vec{C}\cdot\vec{E}^{LR}|\leq\left[\sin\left(\frac{\pi}{2M}\right)\right]^{-N}\cos\left(\frac{\pi}{2M}\eta+\Phi_1+...+\Phi_N\right), \qquad (15)$$

where the cosine comes from the phases of the sums in (11). These phases can be found from the definition (12). As only vectors rotated by a multiple of $\frac{\pi}{M}$ are summed (or subtracted) in (12), each phase Φ_n can acquire only a restricted set of values. Namely:

$$\Phi_n=\begin{cases}\frac{\pi}{2M}+\frac{\pi}{M}k & \text{for } M \text{ even},\\ \frac{\pi}{M}k & \text{for } M \text{ odd},\end{cases} \qquad (16)$$

with $k = 0, ..., 2M-1$, i.e. for M even, Φ_n is an odd multiple of $\frac{\pi}{2M}$; and for M odd, Φ_n is an even multiple of $\frac{\pi}{2M}$. Thus, the sum $\Phi_1 + ... + \Phi_N$ is an even multiple of $\frac{\pi}{2M}$, except for M even and N odd. Keeping in mind the definition of η, given in (7), one finds the argument of $\cos\left(\frac{\pi}{2M}\eta + \Phi_1 + ... + \Phi_N\right)$ is always odd multiple of $\frac{\pi}{2M}$, which implies the maximum value of the cosine is equal to $\cos\left(\frac{\pi}{2M}\right)$. Finally, the multisetting Bell inequality reads:

$$|\vec{C} \cdot \vec{E}^{LR}| \leq \left[\sin\left(\frac{\pi}{2M}\right)\right]^{-N} \cos\left(\frac{\pi}{2M}\right). \tag{17}$$

This inequality, when reduced to two parties choosing between two settings each, recovers the famous Clauser-Horne-Shimony-Holt inequality [7]. For higher number of parties, still choosing between two observables, it reduces to tight two-setting inequalities [8]. When N observers choose between three observables the inequalities of Żukowski and Kaszlikowski are obtained [15], and for continuous range of settings ($M \to \infty$) it recovers the inequality of Żukowski [6].

III. QUANTUM VIOLATIONS

In this section we present a Bell operator associated with the inequality (17). Next, it is used to derive the necessary and sufficient condition for the violation of the inequality. Using this condition we recover already known results and present some new ones.

The form of the coefficients $c_{m_1...m_N} = \cos(\phi^1_{m_1} + ... + \phi^N_{m_N})$ we have chosen is exactly the same as the quantum correlation function $E^{GHZ}_{m_1...m_N} = \cos(\phi^1_{m_1} + ... + \phi^N_{m_N})$ for the Greenberger-Horne-Zeilinger state:

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\Big[|0\rangle_1...|0\rangle_N + |1\rangle_1...|1\rangle_N\Big], \tag{18}$$

where the vectors $|0\rangle_n$ and $|1\rangle_n$ are the eigenstates of local σ^n_z operator of the nth party. For this state the two vectors $\vec{C}$ and $\vec{E}^{GHZ}$ are equal (thus parallel), which means that the state $|\psi^+\rangle$ maximally violates inequality (17). The value of the left hand side of (17) is given by the scalar product of $\vec{E}^{GHZ}$ with itself:

$$\vec{E}^{GHZ} \cdot \vec{E}^{GHZ} = \sum_{m_1,...,m_N=0}^{M-1} \cos^2(\phi^1_{m_1} + ... + \phi^N_{m_N}). \tag{19}$$

Using the trigonometric identity $\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$ one can rewrite this expression into the form:

$$\vec{E}^{GHZ} \cdot \vec{E}^{GHZ} = \frac{1}{2}M^N + \frac{1}{2}\sum_{m_1,...,m_N=0}^{M-1} \cos[2(\phi^1_{m_1} + ... + \phi^N_{m_N})]. \tag{20}$$

As before, the second term can be written as a real part of a complex number. Putting the values of angles (6) one arrives at:

$$\frac{1}{2}\mathrm{Re}\left(\exp\left(i\frac{\pi}{M}\eta\right)\prod_{n=1}^{N}\sum_{m_n=0}^{M-1}\exp\left(i\frac{2\pi}{M}m_n\right)\right). \tag{21}$$

Note that $e^{i\frac{2\pi}{M}}$ is a primitive complex Mth root of unity. Since all complex roots of unity sum up to zero the above expression vanishes, and a maximal quantum value of the left hand side of (17) equals:

$$\vec{E}^{GHZ} \cdot \vec{E}^{GHZ} = \frac{1}{2}M^N. \tag{22}$$

If instead of $|\psi^+\rangle$ one chooses the state $|\psi^-\rangle = \frac{1}{\sqrt{2}}[|0\rangle_1...|0\rangle_N - |1\rangle_1...|1\rangle_N]$, for which the correlation function is given by $E^{GHZ-}_{m_1...m_N} = -\cos(\phi^1_{m_1} + ... + \phi^N_{m_N})$, one arrives at a minimal value of the Bell expression, equal to $-\frac{1}{2}M^N$, as the vectors $\vec{C}$ and $\vec{E}^{GHZ-}$ are exactly opposite. Since we take a modulus in the Bell expression, both states lead to the same violation.

The Bell operator associated with the Bell expression (17) is defined as:

$$\mathcal{B}' \equiv \sum_{m_1...m_N=0}^{M-1} c_{m_1...m_N} \vec{m}_1 \cdot \vec{\sigma}^1 \otimes ... \otimes \vec{m}_N \cdot \vec{\sigma}^N. \tag{23}$$

Its average in the quantum state ρ is equal to the quantum prediction of the Bell expression, for this state. We shall prove that it has only two eigenvalues $\pm\frac{1}{2}M^N$, and thus is of the simple form:

$$\mathcal{B} \equiv \mathcal{B}(N,M) = \frac{1}{2}M^N\left[|\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-|\right]. \tag{24}$$

Both operators $\mathcal{B}$ and $\mathcal{B}'$ are defined in the Hilbert-Schmidt space with the trace scalar product. To prove their equivalence one should check if the conditions:

$$\mathrm{Tr}(\mathcal{B}'\mathcal{B}) = \mathrm{Tr}(\mathcal{B}\mathcal{B}) = \mathrm{Tr}(\mathcal{B}'\mathcal{B}'), \tag{25}$$

are satisfied. Geometrically speaking, these conditions mean that the "length" and "direction" of the operators are the same.

The trace $\mathrm{Tr}(\mathcal{B}'\mathcal{B})$ involves the traces $\mathrm{Tr}\left(|\psi^\pm\rangle\langle\psi^\pm|\vec{m}_1 \cdot \vec{\sigma}^1 \otimes ... \otimes \vec{m}_N \cdot \vec{\sigma}^N\right)$, which are the quantum correlation functions (averages of the product of local observables) for the GHZ states, and thus are given by $\pm\cos(\phi^1_{m_1} + ... + \phi^N_{m_N})$. Their difference doubles the cosine, which is then multiplied by the same cosine coming from the coefficients $c_{m_1...m_N}$. Thus the main trace takes the form:

$$\mathrm{Tr}(\mathcal{B}\mathcal{B}') = M^N \sum_{m_1...m_N=0}^{M-1} \cos^2(\phi^1_{m_1} + ... + \phi^N_{m_N}) = \frac{1}{2}M^{2N}, \tag{26}$$

where the last equality sign follows from the considerations below Eq. (19).

The middle trace of (25) is given by $\mathrm{Tr}(\mathcal{B}\mathcal{B}) = \frac{1}{2}M^{2N}$, which directly follows from the orthonormality of the states $|\psi^{\pm}\rangle$.

The last trace of (25) is more involved. Inserting decomposition (23) into $\mathrm{Tr}(\mathcal{B}'\mathcal{B}')$ gives:

$$\sum_{\substack{m_1...m_N,\\ m'_1...m'_N=0}}^{M-1} \cos(\phi^1_{m_1}+...+\phi^N_{m_N})\cos(\phi^1_{m'_1}+...+\phi^N_{m'_N})$$
$$\times \mathrm{Tr}[(\vec{m}_1\cdot\vec{\sigma}^1)(\vec{m}'_1\cdot\vec{\sigma}^1)]...\mathrm{Tr}[(\vec{m}_N\cdot\vec{\sigma}^N)(\vec{m}'_N\cdot\vec{\sigma}^N)]$$

The local traces are given by:

$$\mathrm{Tr}[(\vec{m}_n\cdot\vec{\sigma}^n)(\vec{m}'_n\cdot\vec{\sigma}^n)] = 2\vec{m}_n\cdot\vec{m}'_n = 2\cos(\phi^n_{m_n}-\phi^n_{m'_n}). \tag{27}$$

Thus, the factor 2^N appears in front of the sums. We write all the cosines (of sums and differences) in terms of individual angles, insert these decompositions into $\mathrm{Tr}(\mathcal{B}'\mathcal{B}')$, and perform all the multiplications. Note that whenever the final product term involves at least one expression like $\cos\phi^n_{m_n}\sin\phi^n_{m_n} = \frac{1}{2}\sin(2\phi^n_{m_n})$ (or for the primed angles) its contribution to the trace vanish after the summations [for the reasons discussed in Eq. (21)]. Moreover, in the decomposition of $\cos(\phi^n_{m_n}-\phi^n_{m'_n}) = \cos\phi^n_{m_n}\cos\phi^n_{m'_n}+\sin\phi^n_{m_n}\sin\phi^n_{m'_n}$ only the products of the same trigonometric functions appear. In order to contribute to the trace they must be multiplied by again the same functions. Since the decompositions of cosines of sums only differ in angles (primed or unprimed) and not in the individual trigonometric functions, the only contributing terms come from the product of exactly the same individual trigonometric functions in the decomposition of $\cos(\phi^1_{m_1}+...+\phi^N_{m_N})$ and $\cos(\phi^1_{m'_1}+...+\phi^N_{m'_N})$. There are 2^{N-1} such products, as many as the number of terms in the decomposition of $\cos(\phi^1_{m_1}+...+\phi^N_{m_N})$. Each product involves $2N$ squared individual trigonometric functions. Each of these functions can be written in terms of cosines of a double angle, e.g. $\sin^2\phi^n_{m_n} = \frac{1}{2}(1-\cos(2\phi^n_{m_n}))$, and the last cosine does not contribute to the sum [again due to (21)]. Finally the trace reads:

$$\mathrm{Tr}(\mathcal{B}'\mathcal{B}') = 2^N \sum_{\substack{m_1...m_N,\\ m'_1...m'_N=0}}^{M-1} 2^{N-1}\frac{1}{2^{2N}} = \frac{1}{2}M^{2N}. \tag{28}$$

Thus, equations (25) are all satisfied, i.e. both operators $\mathcal{B}$ and $\mathcal{B}'$ are equal. Only the states which have contributions in the subspace spanned by $|\psi^{\pm}\rangle$ can violate the inequality (17).

Necessary and sufficient condition for the violation of the inequality. The expected quantum value of the Bell expression, using Bell operator, reads:

$$\mathrm{Tr}(\mathcal{B}(N,M)\rho) = \frac{M^N}{2}\left[\mathrm{Tr}(|\psi^+\rangle\langle\psi^+|\rho) - \mathrm{Tr}(|\psi^-\rangle\langle\psi^-|\rho)\right]. \tag{29}$$

The violation condition is obtained after maximization, for a given state, over the position of the xy plane, in which the observables lie.

An arbitrary state (density operator) of N qubits can be decomposed using local Pauli operators as:

$$\rho = \frac{1}{2^N}\sum_{\mu_1...\mu_N=0}^{3} T_{\mu_1...\mu_N}\sigma_{\mu_1}\otimes...\otimes\sigma_{\mu_N}, \tag{30}$$

where the set of averages $T_{\mu_1...\mu_N} = \mathrm{Tr}[\rho(\sigma_{\mu_1}\otimes...\otimes\sigma_{\mu_N})]$ forms the so-called correlation tensor. The correlation tensors of the projectors $|\psi^{\pm}\rangle\langle\psi^{\pm}|$ are denoted by $T^{\pm}_{\nu_1...\nu_N}$. Using the linearity of the trace operation and the fact that the trace of the tensor product is given by the product of local traces, one can write $\mathrm{Tr}(|\psi^{\pm}\rangle\langle\psi^{\pm}|\rho)$ in terms of correlation tensors:

$$\frac{1}{2^{2N}}\sum_{\substack{\mu_1...\mu_N,\\ \nu_1...\nu_N=0}}^{3} T^{\pm}_{\nu_1...\nu_N}T_{\mu_1...\mu_N}\mathrm{Tr}(\sigma_{\mu_1}\sigma_{\nu_1})...\mathrm{Tr}(\sigma_{\mu_N}\sigma_{\nu_N}).$$

Since each of the N local traces $\mathrm{Tr}(\sigma_{\mu_n}\sigma_{\nu_n}) = 2\delta_{\mu_n\nu_n}$, the global trace is given by:

$$\mathrm{Tr}(|\psi^{\pm}\rangle\langle\psi^{\pm}|\rho) = \frac{1}{2^N}\sum_{\mu_1...\mu_N=0}^{3} T^{\pm}_{\mu_1...\mu_N}T_{\mu_1...\mu_N}. \tag{31}$$

The nonvanishing correlation tensor components of the GHZ states $|\psi^{\pm}\rangle$ are the same in the z plane: $T^{\pm}_{z...z} = 1$ for N even; and are exactly opposite in the xy plane: $T^{+}_{i_1...i_N} = -T^{-}_{i_1...i_N} = (-1)^{\xi}$ with 2ξ indices equal to y and all remaining equal to x. Inserting the traces (31) into the averaged Bell operator (29) one finds that the components in the z plane cancel out, and components in the xy plane double themselves. Finally, the necessary and sufficient condition to satisfy the inequality is given by:

$$\left(\frac{M}{2}\right)^N \max \sum_{i_1...i_N\in I_\xi} (-1)^{\xi} T_{i_1...i_N} \le B_{LR}(N,M), \tag{32}$$

where the maximization is performed over the choice of local coordinate systems, I_ξ includes all sets of indices $i_1...i_N$ with 2ξ indices equal to y and the rest equal to x, and

$$B_{LR}(N,M) = \left[\sin\left(\frac{\pi}{2M}\right)\right]^{-N}\cos\left(\frac{\pi}{2M}\right) \tag{33}$$

denotes the local realistic bound.

We now present examples of states, which violate the new inequality. As a measure of violation, $V(N,M)$, we take the average (quantum) value of the Bell operator in a given state, divided by the local realistic bound:

$$V(N,M) = \frac{\langle\mathcal{B}(N,M)\rangle_\rho}{B_{LR}(N,M)}. \tag{34}$$

GHZ state. First, let us simply consider $|\psi^{\pm}\rangle$. For the case of two settings per side one recovers previously known results [8, 16, 17]:

$$V(N,2) = 2^{(N-1)/2}. \tag{35}$$

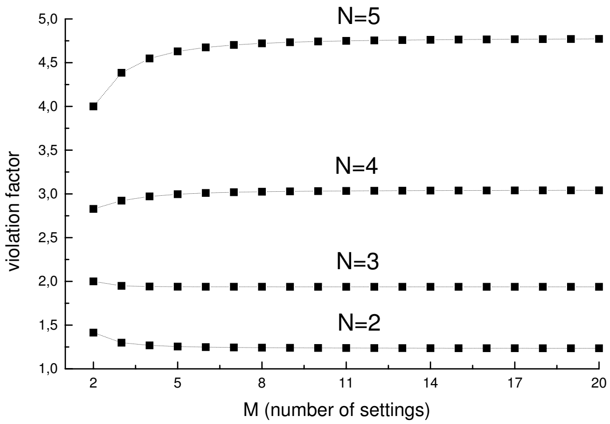

FIG. 1: Violation factor as a function of number of local settings, M, for the N-qubit GHZ state.

For three settings per side the result of Żukowski and Kaszlikowski is obtained [15]:

$$V(N,3) = \frac{1}{\sqrt{3}}\left(\frac{3}{2}\right)^N. \quad (36)$$

For the continuous range of settings one recovers [6]:

$$V(N,\infty) = \frac{1}{2}\left(\frac{\pi}{2}\right)^N. \quad (37)$$

In the intermediate (unexplored before) regime one has:

$$V(N,M) = \frac{1}{2\cos\left(\frac{\pi}{2M}\right)}\left(M\sin\left(\frac{\pi}{2M}\right)\right)^N. \quad (38)$$

For a fixed number of parties $N > 3$ the violation increases with the number of local settings. Surprisingly, the inequality implies for the cases of $N = 2$ and $N = 3$ that the violation decreases when the number of local settings grows. This behaviour is shown in the Fig. 1. The violation of local realism always grows with increasing number of parties.

Generalized GHZ state. Consider the GHZ state with free real coefficients:

$$|\psi\rangle = \cos\alpha|0\rangle_1...|0\rangle_N + \sin\alpha|1\rangle_1...|1\rangle_N. \quad (39)$$

Its correlation tensor in the xy plane has the following nonvanishing components: $T_{x...x} = \sin 2\alpha$, and the components with 2ξ indices equal to y and the rest equal to x take the value of $(-1)^\xi \sin 2\alpha$ (there are $2^{N-1} - 1$ such components). Thus, all 2^{N-1} terms contribute to the violation condition (32). The violation factor is equal to $V(N,M) = \frac{M^N}{2B_{LR}(N,M)}\sin 2\alpha$. For $N > 3$ and $M > 2$ the violation is bigger than the violation of standard two-setting inequalities [17]. Moreover, some of the states $|\psi\rangle$, for small α and odd N, do not violate *any* two-settings correlation function Bell inequality [10], and violate the multisetting inequality.

Bound entangled state. Interestingly, the inequality can reveal non-classical correlations of a bound entangled state introduced by Dür [11]:

$$\rho_N = \frac{1}{N+1}\left(|\phi\rangle\langle\phi| + \frac{1}{2}\sum_{k=1}^N (P_k + \tilde{P}_k)\right), \quad (40)$$

with $|\phi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle_1...|0\rangle_N + e^{i\alpha_N}|1\rangle_1...|1\rangle_N\right]$ (α_N is an arbitrary phase), and P_k being a projector on the state $|0\rangle_1...|1\rangle_k...|0\rangle_N$ with "1" on the kth position ($\tilde{P}_k$ is obtained from P_k after replacing "0" by "1" and vice versa). As originally shown in [11] this state violates Mermin-Klyshko inequalities for $N \geq 8$. The new inequality predicts the violation factor of:

$$V(N,M) = \frac{1}{N+1}\frac{M^N \ \cos\alpha_N}{2B_{LR}(N,M)}, \quad (41)$$

which comes from the contribution of the GHZ-like state $|\phi\rangle$ to the bound entangled state. One can follow Ref. [18] and change the Bell-operator (24) such that the state $|\phi\rangle$ becomes its eigenstate. The new operator, $\tilde{\mathcal{B}}(N,M)$, is obtained after applying local unitary transformations $U = |0\rangle\langle 0| + e^{i\alpha_N/N}|1\rangle\langle 1|$ to the operator (24), i.e. $\tilde{\mathcal{B}}(N,M) = U^{\otimes N}\mathcal{B}U^{\dagger\otimes N}$. The violation factor of the new inequality is higher than (41), and equal to:

$$\tilde{V}(N,M) = \frac{1}{N+1}\frac{M^N}{2B_{LR}(N,M)}. \quad (42)$$

If one sets $M = 3$ it appears that the number of parties sufficient to see the violation (42) reduces to $N \geq 7$ [18]. On the other hand the result of [19] shows that the infinite range of settings further reduces the number of parties to $N \geq 6$. Using the new inequality, $M = 5$ settings per side suffice to already violate local realism with $N \geq 6$ parties.

IV. POSITIVE PARTIAL TRANSPOSE

In this section it is shown that all the states with positive partial transpose with respect to all subsystems satisfy the multisetting inequality (17). This result further supports the conjecture by Peres, that all such states can admit local realistic description [14]. First we briefly review partial transpositions, next present an inequality that all such states must satisfy, and finally compare it with the Bell inequality (17).

The partial transpose of an operator on a Hilbert space $H_1 \otimes H_2$ is defined by:

$$\left(\sum_l A_l^1 \otimes A_l^2\right)^{T_1} = \sum_l A_l^{1\,T} \otimes A_l^2, \quad (43)$$

where the superscript T denotes transposition in the given basis. The positivity of partial transpose is found to be a necessary condition for separability [12, 13]. The operator obtained by the partial transpose of any separable state is positive (PPT - positive partial transpose). In the bipartite case of two qubits or qubit-qutrit system, the PPT criterion is also sufficient for separability.

In the multipartite case the situation complicates as one can have many different partitions into set of particles, for example four particle system 1234 can be split e.g. into $12-34$ or $1-2-34$. Suppose one splits N particles into p groups, take as an example the split into three groups $1-2-34$. The state is called p-PPT if it has positive *all possible* partial transposes. Fortunately, positivity of partial transpose with respect to certain set of subsystems is the same as positivity with the respect to all remaining subsystems. In the example one should check the positivity of operator obtained after transposition of subsystem 1, next 2, and finally 34.

All the p-PPT states were recently shown to satisfy the following inequalities [20]:

$$\mathrm{Tr}\left[\left(|\psi^{\pm}\rangle\langle\psi^{\pm}| - (1-2^{2-p})|\psi^{\mp}\rangle\langle\psi^{\mp}|\right)\rho\right] \leq 2^{1-p}, \quad (44)$$

i.e. if $|\psi^{+}\rangle$ appears in the first term within the trace, $|\psi^{-}\rangle$ appears in the second term, and vice versa. Omitting the positive factor $2^{2-p}\mathrm{Tr}\,(|\psi^{\mp}\rangle\langle\psi^{\mp}|\rho)$ one arrives at the Bell operator form:

$$\left|\mathrm{Tr}\left[\left(|\psi^{+}\rangle\langle\psi^{+}| - |\psi^{-}\rangle\langle\psi^{-}|\right)\rho\right]\right| \leq 2^{1-p}. \quad (45)$$

Following the conjecture by Peres let us put $p=N$. We shall show that if inequality (45) is satisfied, with $p=N$, then also the Bell inequality (17) is not violated. Using the form of the Bell operator (24) the upper bound of the Bell inequality, for N-PPT states, is found to read:

$$|\mathrm{Tr}(\mathcal{B}(N,M)\rho_{N-\mathrm{PPT}})| \leq \left(\frac{M}{2}\right)^N, \quad (46)$$

and it can never reach the local realistic bound $B_{LR}(M,N)$. This is shown using the violation factor:

$$V_{N-\mathrm{PPT}}(N,M) = \left(\frac{M}{2}\right)^N \frac{[\sin\left(\frac{\pi}{2M}\right)]^N}{\cos\left(\frac{\pi}{2M}\right)}. \quad (47)$$

Since $\sin\left(\frac{\pi}{2M}\right) \leq \frac{\pi}{2M}$ and $\cos\left(\frac{\pi}{2M}\right) \geq \frac{1}{\sqrt{2}}$, where we have put $M=2$ as a minimal amount of settings for which Bell inequality makes sense, the violation factor is bounded by $V_{N-\mathrm{PPT}}(N,M) \leq \sqrt{2}(\pi/4)^N$. The simplest system on which one can perform partial transposes consists of $N=2$ particles, thus $V_{N-\mathrm{PPT}}(N,M) \leq \sqrt{2}(\pi/4)^2 \simeq 0.87$. None of the N-PPT states violates the Bell inequality (17). It is worth mentioning that for $M=2$ setting case the violation $V_{N-\mathrm{PPT}}(N,2) = 2^{(1-N)/2}$ confirms the results of Werner and Wolf [16], who gave the conjecture of Peres a sharp mathematical form [21].

V. COMMUNICATION COMPLEXITY

Bell inequalities describe a performance of quantum communication complexity protocols [5]. In this section we follow this general link and present communication complexity problems associated with the inequality (17). It is proven that the quantum protocol outperforms the best classical protocol for arbitrary number of parties and observables.

In the communication complexity problems (CCP) one studies the information exchange between participants *locally* performing computations, in order to accomplish a *globally* defined task [22]. Let us focus on a variant of a CCP, in which each of N separated partners receives arguments, $y_n = \pm 1$ and $x_n = 0,...,M-1$, of some globally defined function, $\mathcal{F} \equiv \mathcal{F}(y_1, x_1, ..., y_N, x_N)$. The y_n inputs are assumed to be randomly distributed, and x_n inputs can in general be distributed according to a weight $\mathcal{W}(x_1,...,x_2)$. The goal is to maximize the probability that Alice arrives at the correct value of the function, under the restriction that $N-1$ bits of overall communication are allowed. Before participants receive their inputs they are allowed to do anything from which they can derive benefit. In particular, they can share some correlated strings of numbers in the classical scenario or entangled states in the quantum case.

The problem. Following [5] one chooses for a task-function:

$$\mathcal{F} \;=\; y_1...y_N \mathrm{Sign}[\cos(\phi^1_{x_1} + ... + \phi^N_{x_N})] = \pm 1, \quad (48)$$

with the angles defined by (6). According to the angles definition the cosine can never be zero, so the problem is well-defined for all N and M. Additionally, the x_n inputs are distributed with the weight:

$$\mathcal{W}(x_1,...,x_2) = (1/\mathcal{N})|\cos(\phi^1_{x_1} + ... + \phi^N_{x_N})|, \quad (49)$$

where the normalization factor is given by $\mathcal{N} = \sum_{x_1...x_N=0}^{M-1} |\cos(\phi^1_{x_1} + ... + \phi^N_{x_N})|$. After the communication takes place, if Alice misses some of the random variables y_n, her "answer" can only be random. Thus, in an optimal protocol each party must communicate one bit. There are only two communication structures which lead to a non-random answer: (i) a star – each party transmits one bit directly to Alice, and (ii) a chain – sequence of a peer-to-peer exchanges with Alice at the end. The task is to maximize the probability of correct answer $\mathcal{A} \equiv \mathcal{A}(y_1, x_1, ..., y_N, x_N)$. Since both $\mathcal{A}$ and $\mathcal{F}$ are dichotomic variables this amounts in maximizing:

$$P_{\mathrm{correct}} = \frac{1}{2^N} \sum_{\mathbf{y},\mathbf{x}} \mathcal{W}(x_1,...,x_2) P_{\mathbf{y},\mathbf{x}}(\mathcal{AF}=1), \quad (50)$$

where $\frac{1}{2^N}$ describes (random) distribution of y_n's, and $P_{\mathbf{y},\mathbf{x}}(\mathcal{AF}=1)$ is a probability that $\mathcal{A}=\mathcal{F}$ for given inputs $\mathbf{y} \equiv (y_1,...,y_N)$ and $\mathbf{x} \equiv (x_1,...,x_N)$. It is useful to express the last probability in terms of an average value of a product $\langle\mathcal{AF}\rangle_{\mathbf{y},\mathbf{x}}$, i.e. $P_{\mathbf{y},\mathbf{x}}(\mathcal{AF}=1) = \frac{1}{2}[1 + \langle\mathcal{AF}\rangle_{\mathbf{y},\mathbf{x}}]$.

Since $\mathcal{F}$ is independent of $\mathcal{A}$, and for given inputs it is constant, one has $P_{\mathbf{y},\mathbf{x}}(\mathcal{A}\mathcal{F}=1)=\frac{1}{2}[1+\mathcal{F}\langle\mathcal{A}\rangle_{\mathbf{y},\mathbf{x}}]$. Finally the probability of correct answer reads $P_{\text{correct}}=\frac{1}{2}[1+(\mathcal{F},\mathcal{A})]$, and it is in one-to-one correspondence with a "weighted" scalar product (average success):

$$(\mathcal{F},\mathcal{A})=\frac{1}{2^N}\sum_{\mathbf{y},\mathbf{x}}\mathcal{W}(x_1,...,x_2)\mathcal{F}\langle\mathcal{A}\rangle_{\mathbf{y},\mathbf{x}}. \tag{51}$$

Using the definitions (49) for $\mathcal{W}$ and (48) for $\mathcal{F}$ one gets:

$$(\mathcal{F},\mathcal{A})=\frac{1}{2^N}\frac{1}{\mathcal{N}}\sum_{\mathbf{y},\mathbf{x}}y_1...y_N\cos(\phi^1_{x_1}+...+\phi^N_{x_N})\langle\mathcal{A}\rangle_{\mathbf{y},\mathbf{x}}, \tag{52}$$

with angles given by (6). We focus our attention on maximization of this quantity.

Classical scenario. In the *best* classical protocol each party locally computes a bit function $e_n=y_nf(x_n,\lambda)$, with $f(x_n,\lambda)=\pm1$, where λ denotes some previously shared classical resources. Next, the bit is sent to Alice, who puts as an answer the product $\mathcal{A}_c=y_1f(x_1,\lambda)e_2...e_N=y_1...y_Nf(x_1,\lambda)...f(x_N,\lambda)$. The same answer can be reached in the chain strategy, simply the nth party sends $e_n=y_nf(x_n,\lambda)e_{n-1}$. For the given inputs the procedure is always the same, i.e. $\langle\mathcal{A}_c\rangle_{\mathbf{y},\mathbf{x}}=\mathcal{A}_c$. To prove the optimality of this protocol, one follows the proof of Ref. [23], with the only difference that x_n is a M-valued variable now. This, however, does not invalidate any of the steps of [23], and we will not repeat that proof.

Inserting the product form of $\mathcal{A}_c$ into the average success (52), using the fact that $y_n^2=1$, and summing over all y_n's one obtains:

$$(\mathcal{F},\mathcal{A}_c)=\frac{1}{\mathcal{N}}\sum_{x_1...x_N=0}^{M-1}\cos(\phi^1_{x_1}+...+\phi^N_{x_N})f(x_1,\lambda)...f(x_N,\lambda), \tag{53}$$

which has the same structure as local realistic expression (8). Thus, the highest classically achievable average success is given by a local realistic bound: $\max(\mathcal{F},\mathcal{A})=(1/\mathcal{N})B_{LR}(N,M)$.

Quantum scenario. In the quantum case participants share a N-party entangled state ρ. After receiving inputs each party measures x_nth observable on the state, where the observables are enumerated as in the Bell inequality (17). This results in a measurement outcome, f_n. Each party sends $e_n=y_nf_n$ to Alice, who then puts as an answer a product $\mathcal{A}_q=y_1...y_Nf_1...f_N$. For the given inputs the average answer reads $\langle\mathcal{A}_q\rangle_{\mathbf{y},\mathbf{x}}=y_1...y_N\langle f_1...f_N\rangle=y_1...y_NE^{\rho}_{x_1...x_N}$, and the maximal average success is given by a quantum bound of:

$$(\mathcal{F},\mathcal{A}_q)=\frac{1}{\mathcal{N}}\sum_{x_1...x_N=0}^{M-1}\cos(\phi^1_{x_1}+...+\phi^N_{x_N})E^{\rho}_{x_1...x_N}. \tag{54}$$

The average advantage of quantum versus classical protocol can be quantified by a factor $(\mathcal{F},\mathcal{A}_q)/(\mathcal{F},\mathcal{A}_c)$ which is equal to a violation factor, $V(N,M)$, introduced before. Thus, all the states which violate the Bell inequality (including bound entangled state) are a useful resource for the communication complexity task. Optimally one should use the GHZ states $|\psi^{\pm}\rangle$, as they maximally violate the inequality.

$N\backslash M$	2	3	4	5	∞
2	1.1381	1.1196	1.1009	1.1002	1.0909
3	1.3333	1.2919	1.2815	1.2773	1.2709
4	1.3657	1.4395	1.4038	1.4258	1.4192
5	1.6000	1.5582	1.5467	1.5418	1.5336

TABLE I: The ration between probabilities of success in quantum and classical case $P^{QM}_{\text{correct}}/P^{cl}_{\text{correct}}$ for the communication complexity problem with N observers and M settings. Quantum protocol uses GHZ state.

Alternatively, one can compare the probabilities of success, P_{correct}, in quantum and classical case. Clearly, one outperforms classical protocols for every N and every M. As an example, in Table I we gather the ratios between quantum and classical success probabilities for small number of participants.

One can ask about a CCP with no random inputs y_n. Since the numbers x_n already represent $\lg M$ bits of information, and only one bit can be communicated, this looks like a plausible candidate for a quantum advantage. However, in such a case a classical answer cannot be put as a product of outcomes of local computations (compare [23]), and thus there is no Bell inequality which would describe the best classical protocol. Since classical performance of *all* CCPs which can lead to quantum advantage is given by some Bell inequality [5], the task without y_n's cannot lead to quantum advantage.

VI. SUMMARY

We presented a multisetting Bell inequality, which unifies and generalizes many previous results. Examples of quantum states which violate the inequality were given. It was also proven that all the states with positive partial transposes with respect to all subsystems cannot violate the inequality. Finally, the states which violate it were shown to reduce the communication complexity of computation of certain globally defined function. The Bell inequality presented is the only inequality which incorporates arbitrary number of settings for arbitrary number of observers making measurements on two-level systems, to date.

Acknowledgments

We thank M. Żukowski for valuable discussions. W.L. and T.P. are supported by Foundation for Polish Science and MNiI Grant no. 1 P03B 049 27. The work is part of the VI-th EU Framework programme QAP (Qubit Applications) Contract No. 015848.

[1] J. S. Bell, Physics **1**, 195 (1964).
[2] J. F. Clauser and A. Shimony, Rep. Prog. Phys. **41**, 1881 (1978); D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990); T. Paterek, W. Laskowski, and M. Żukowski, Mod. Phys. Lett. A **21**, 111 (2006).
[3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[4] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991); V. Scarani and N. Gisin, Phys. Rev. Lett. **87**, 117901 (2001); A. Acin, N. Gisin, and V. Scarani, Quant. Inf. Comp. **3**, 563 (2003).
[5] Č. Brukner, M. Żukowski, J.-W. Pan, and A. Zeilinger, Phys. Rev. Lett. **92**, 127901 (2004).
[6] M. Żukowski, Phys. Lett. A **177**, 290 (1993).
[7] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[8] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990); M. Ardehali, Phys. Rev. A **46**, 5375 (1992); A. V. Belinskii and D. N. Klyshko, Phys. Usp. **36**, 653 (1993).
[9] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, The Netherlands, 1989).
[10] M. Żukowski, Č. Brukner, W. Laskowski, and M. Wieśniak, Phys. Rev. Lett. **88**, 210402 (2002).
[11] W. Dür, Phys. Rev. Lett. **87**, 230402 (2001).
[12] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[13] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
[14] A. Peres, Found. Phys, **29**, 589 (1999).
[15] M. Żukowski and D. Kaszlikowski, Phys. Rev. A **56**, R1682 (1997).
[16] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001).
[17] M. Żukowski and Č. Brukner, Phys. Rev. Lett. **88**, 210401 (2002).
[18] D. Kaszlikowski, L. C. Kwek, J. Chen, and C. H. Oh, Phys. Rev. A **66**, 052309 (2002).
[19] A. Sen (De), U. Sen, and M. Żukowski, Phys. Rev. A **66**, 062318 (2002).
[20] K. Nagata, Phys. Rev. A **66**, 064101 (2002).
[21] R. F. Werner and M. M. Wolf, Phys. Rev. A **61**, 062102 (2000).
[22] A. C.-C. Yao, in *Proceedings of the 11th Annual ACM Symposium on Theory of Computing* (ACM Press, New York, 1979).
[23] P. Trojek, C. Schmid, M. Bourennane, Č. Brukner, M. Żukowski, and H. Weinfurter, Phys. Rev. A **72**, 050305(R) (2005).